Persistent algebras

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How to endow persistent modules with an algebra structure.
1 Persistent modules
2 Homology and cup product
3 Extension of the cup product to a whole module
This story start with a point cloud

Figure – A set of points with some underlying interesting topology.
Let $X_t$ be a filtration...
A bit of (co)homological magic wand

\[ H^*(X_0) \xrightarrow{\rho_0^1} H^*(X_1) \xrightarrow{\rho_1^2} H^*(X_2) \xrightarrow{\rho_2^3} H^*(X_3) \xrightarrow{\rho_3^4} \ldots \]

**Figure** – A persistence module on \( \mathbb{N} \)
The module structure

Module structure over $k[x]$: 

Let 

$$M = \bigoplus_{n \in \mathbb{N}} H^*(X_n)$$

and define the multiplication by elements of $k[x]$ by 

$$\forall u \in M, u \in H^*(X_n),$$

$$x.u = \rho_n^{n+1}(u)$$
Module structure over $k[x]$:

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and define the multiplication by elements of $k[x]$ by

$$\forall u \in M, u \in H^*(X_n),$$

$$x.u = \rho_{n+1}^n(u)$$
We can build a bifiltration on $\mathbb{R}^2$:

$$\forall x' \geq x, y' \geq y, X_{x,y} \subseteq X_{(x',y')}.$$ 

and apply the (co)homology functor.
Persistent modules

Multimodules

Bimodule $X_{x,y} = f^{-1}([x, y])$

$$H^*(X_{x,y}) \quad H^*(X_{x',y})$$

$$\rho = H^*(\subseteq)$$

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The Theory of Multidimensional Persistence
Gunnar Carlsson & Afra Zomorodian
### Singular homology

#### Singular simplex

- $X$ a topological. $\Delta^n$ the $n$ standard simplex.
- A *singular $n$-simplex* is a continuous map $\sigma : \Delta^n \to X$.

#### Chains

$C_n(X)$ free abelian group with basis the singular $n$-simplices.

#### Cochains

$C^n(X) = \text{Hom}(C_n(X), k)$. 
The cochains $C^n(X) = \text{Hom}(C_n(X), k)$ form a chain complex

$$\delta^n : C^n(X) \to C^{n+1}(X)$$

with

$$\delta \varphi(\sigma) = \sum_i (-1)^i \varphi(\sigma|\{v_0, \ldots, \hat{v}_i, \ldots, v_k\})$$

The $n$-th cohomology group is $H^n(X) = \text{Ker} \delta^n / \text{Im} \delta^{n-1}$.

**Homology module**

$$H^*(X) = \bigoplus_{i \in \mathbb{N}} H^i(X)$$
Cup product

Figure – Artistic representation of a cup product
Let $\varphi \in C^k(X)$ and $\psi \in C^l(X)$.

$$\varphi \smile \psi(\sigma) = \varphi(\sigma|[v_0, \ldots, v_k])\psi(\sigma|[v_k, \ldots, v_{k+l}]) \in C^{k+l}(X).$$

This product induces a product to the cohomological level: the cup product.

For $\alpha \in H^k(X)$ and $\beta \in H^l(X)$,

$$\alpha \smile \beta = (-1)^{kl} \beta \smile \alpha.$$
Cup product

Let \( \varphi \in C^k(X) \) and \( \psi \in C^l(X) \).

\[
\varphi \smile \psi(\sigma) = \varphi(\sigma|[v_0, \ldots, v_k])\psi(\sigma|[v_k, \ldots, v_{k+l}]) \in C^{k+l}(X).
\]

Cup product

This product induces a product to the cohomological level: the cup product.

Graded commutative

For \( \alpha \in H^k(X) \) and \( \beta \in H^l(X) \),

\[
\alpha \smile \beta = (-1)^{kl} \beta \smile \alpha.
\]
Let $\varphi \in C^k(X)$ and $\psi \in C^l(X)$.

$$\varphi \smile \psi(\sigma) = \varphi(\sigma|[v_0, \ldots, v_k])\psi(\sigma|[v_k, \ldots, v_{k+l}]) \in C^{k+l}(X).$$

This product induce a product to the cohomological level: the cup product.

For $\alpha \in H^k(X)$ and $\beta \in H^l(X)$,

$$\alpha \smile \beta = (-1)^{kl}\beta \smile \alpha.$$
The cohomology is a functor $H^* : \text{Top} \to k$-algebra.

$$H^*(X) = \bigoplus_{d \in \mathbb{N}} H^d(X)$$
Extension of the cup product to a whole module

Two bifiltration with same persistent bimodule

\[ S^1 \lor S^1 \lor S^2 \lor S^1 \lor S^1 \]

\[ S^1 \lor S^1 \lor S^1 \lor S^1 \lor S^1 \]

\[ S^1 \lor S^1 \lor S^1 \lor S^1 \lor S^1 \]

\[ S^1 \lor S^1 \lor S^1 \lor S^1 \lor S^1 \]

\[ \text{Figure} – \text{Two topological filtration which give the same persistent bimodule.} \]
Can we extend the cup product to persistence modules?

\[ M = \bigoplus_{(x,y) \in \mathbb{R}^2} \bigoplus_{d \in \mathbb{N}} H^d(X_{x,y}) \]

YES! WE CAN!
Can we extend the cup product to persistence modules?

\[ M = \bigoplus_{(x,y) \in \mathbb{R}^2} \bigoplus_{d \in \mathbb{N}} H^d(X_{x}, y) \]

YES! WE CAN!
Product of two homogeneous vectors

\( \forall t, s \in \mathbb{R}^2 \text{ define } t \vee s = (\max(t_x, s_x), \max(t_y, s_y)) \).

Let \( m \in H^*(X_s), n \in H^*(X_t) \),

\[
m \smile n = (x^{s \vee t - s} m) \smile (x^{s \vee t - t} n)
\]

Cup product in \( M \)

Let \( \alpha = \sum m_s, \forall s, m_s \in M_s, \beta = \sum n_t, \forall t, m_t \in M_t \). The cup product of this two elements is

\[
\alpha \smile \beta = \sum_s \sum_t (x^{s \vee t - s} m_s) \smile (x^{s \vee t - t} n_t)
\]

Proposition :

\( (M, \smile) \) is a graded-commutative (non-unital) ring.
Product of two homogeneous vectors

∀\(t, s \in \mathbb{R}^2\) define \(t \vee s = (\max(t_x, s_x), \max(t_y, s_y))\).

Let \(m \in H^*(X_s), n \in H^*(X_t),\)

\[m \triangleleft n = (x^{s \vee t - s}.m) \triangleright (x^{s \vee t - t}.n)\]

Cup product in \(M\)

Let \(\alpha = \sum m_s, \forall s, m_s \in M_s, \beta = \sum n_t, \forall t, m_t \in M_t.\) The cup product of this two elements is

\[\alpha \triangleleft \beta = \sum_s \sum_t (x^{s \vee t - s}.m_s) \triangleright (x^{s \vee t - t}.n_t)\]

Proposition :

\((M, \triangleleft)\) is a graded-commutative (non-unital) ring.
Product of two homogeneous vectors

∀t, s ∈ ℝ² define \( t ∨ s = (\max(t_x, s_x), \max(t_y, s_y)) \).
Let \( m ∈ H^∗(X_s), n ∈ H^∗(X_t), \)

\[
m ∼ n = (x^{s ∨ t − s}.m) ∼ (x^{s ∨ t − t}.n)
\]

Cup product in M

Let \( α = \sum m_s, ∀s, m_s ∈ M_s, β = \sum n_t, ∀t, m_t ∈ M_t. \) The cup product of this two elements is

\[
α ∼ β = \sum_s \sum_t (x^{s ∨ t − s}.m_s) ∼ (x^{s ∨ t − t}.n_t)
\]

Proposition :

\((M, ∼)\) is a graded-commutative (non-unital) ring.
Figure – Two topological filtration which give the same persistence bimodules, but two different persistent algebra.
Thanks for your attention.