Walking in Poisson Delaunay triangulations

Olivier Devillers

[D. & Hemsley, JoCG:7(1)]
[Chenavier & D., Hal]
[de Castro & D., Hal, DCG]
[D. & Noizet, Hal]
Walking in Delaunay triangulations

Straight walk
Walking in Delaunay triangulations

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Walking in Delaunay triangulations

Straight walk

Exit edge?

One orientation predicate
Walking in Delaunay triangulations

Straight walk

End of walk?

A second orientation predicate
Walking in Delaunay triangulations

Straight walk

Two orientation predicates per edge
Walking in Delaunay triangulations

Visibility walk
Walking in Delaunay triangulations

Visibility walk
Walking in Delaunay triangulations

Visibility walk
Walking in Delaunay triangulations

Visibility walk

3 - 4
Walking in Delaunay triangulations

Visibility walk
Walking in Delaunay triangulations

Visibility walk
Walking in Delaunay triangulations

Visibility walk

Triangle with two exits

One orientation predicate
Walking in Delaunay triangulations

Visibility walk

Triangle with one exit

1.5 orientation predicate

One predicate
if this neighbor tried first

Two predicates
if this neighbor tried first
Walking in Delaunay triangulations

Visibility walk

1.25 orientation predicate per edge?
Walking in Delaunay triangulations

How many edges crossed?

Straight walk

Visibility walk
Walking in Delaunay triangulations

How many edges crossed?

Straight walk

\[ 2n \]

Visibility walk

\[ \infty \geq 2^{3/\sqrt{n}} \text{ randomized} \]

Worst case in a triangulation
Walking in Delaunay triangulations

How many edges crossed?

Straight walk

$2n$

Visibility walk

$2n$

Worst case in a Delaunay triangulation
Walking in Delaunay triangulations

How many edges crossed?

Straight walk

\[ 2n \quad O(\sqrt{n}) \quad \text{[Devroye, Lemaire, & Moreau, 2004]} \]

\[
\frac{64}{3\pi^2} \sqrt{n} + O\left(\frac{1}{\sqrt{n}}\right) \approx 2.16\sqrt{n}
\]

Visibility walk

\[ 2n \quad O(\sqrt{n}) \]

random

Worst case in a Delaunay triangulation
Walking in Delaunay triangulations

Walk between vertices
Walking in Delaunay triangulations

Walk between vertices

Shortest path
Walking in Delaunay triangulations

Walk between vertices

Upper path
Walking in Delaunay triangulations

Walk between vertices

Compass walk
Walking in Delaunay triangulations

Walk between vertices

Voronoi path
Walking in Delaunay triangulations

Walk between vertices

Voronoi path with shortcuts
Walking in Delaunay triangulations

Walk between vertices

Shortest path
Upper path
Compass walk
Voronoi path with shortcuts
Walking in Delaunay triangulations
Walk between vertices

Shortest path  Upper path
Compass walk  Voronoi path  shortcuts
Walking in Delaunay triangulations

Expected length (experiments)

<table>
<thead>
<tr>
<th>Path</th>
<th>Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>Euclidean length</td>
<td>1</td>
</tr>
<tr>
<td>Shortest path</td>
<td>1.04</td>
</tr>
<tr>
<td>Compass walk</td>
<td>1.07</td>
</tr>
<tr>
<td>Shortened V. path</td>
<td>1.16</td>
</tr>
<tr>
<td>Upper path</td>
<td>1.18</td>
</tr>
<tr>
<td>Voronoi path</td>
<td>1.27</td>
</tr>
</tbody>
</table>
## Walking in Delaunay triangulations

<table>
<thead>
<tr>
<th>Path</th>
<th>Expected length (experiments)</th>
<th>theory</th>
</tr>
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<tbody>
<tr>
<td>Euclidean length</td>
<td>1</td>
<td>$\geq 1 + 10^{-11}$</td>
</tr>
<tr>
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<td></td>
</tr>
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<td></td>
</tr>
<tr>
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<td>1.16</td>
<td>1.16 numerical integration</td>
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<td></td>
</tr>
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$\frac{35}{3\pi^2} \approx 1.18$

$\frac{4}{\pi} \approx 1.27$

[Baccelli et al., 2000]
## Walking in Delaunay triangulations

### Expected length (experiments)

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<tr>
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<th>Experiment</th>
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<td>1.27</td>
<td>[Baccelli et al., 2000]</td>
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</tbody>
</table>
Expected length of upper path

Poisson Delaunay triangulation, rate 1

\[ \mathbb{E}[\text{length}] = \mathbb{E} \left[ \sum_{\text{triangle} \in X^3_n} 1_{\text{triangle is Delaunay}} 1_{\text{first edge above } st} \text{length(first edge)} \right] \]
\[ \mathbb{E}\left[\text{length}\right] = \mathbb{E}\left[ \sum_{\text{triangle} \in X_n^3} 1_{\text{triangle is Delaunay}} 1_{\text{first edge above } st} \text{length(first edge)} \right] \]

\[ = n^3 \int_{(\mathbb{R}^2)^3} \mathbb{P}[\text{triangle is Delaunay}] 1_{\text{first edge above } st} \text{length(first edge)} d\text{triangle} \]

**Slivnyak-Mecke**
\[
\mathbb{E}[\text{length}] = \mathbb{E}\left[ \sum_{\text{triangle} \in X^3_n} 1[\text{triangle is Delaunay}] 1[\text{first edge above } st] \text{length(first edge)} \right]
\]

\[
= n^3 \int_{(\mathbb{R}^2)^3} \mathbb{P}[\text{triangle is Delaunay}] 1[\text{first edge above } st] \text{length(first edge)} d\text{triangle}
\]

\[
= n^3 \int_0^\infty \int_0^1 \int_{-r}^{r} e^{-n\pi r^2} \int_{0}^{2\pi} \int_{[0,2\pi]} 2r \sin \frac{\alpha_1 - \alpha_2}{2} r^3 2A(\text{triangle}) d\alpha_1 d\alpha_2 dy_z dx_z dr
\]

Blaschke-Petkantschin
Expected length of upper path

Poisson Delaunay triangulation, rate 1

\[ E[\text{length}] = E\left[ \sum_{\text{triangle} \in X_3^n} 1_{\text{triangle is Delaunay}} 1_{\text{first edge above } st} \text{length(first edge)} \right] \]

\[ = n^3 \int_{(\mathbb{R}^2)^3} \mathbb{P}[\text{triangle is Delaunay}] \mathbb{1}_{\text{first edge above } st} \text{length(first edge)} d\text{triangle} \]

\[ = n^3 \int_0^\infty \int_0^1 \int_{-r}^r \int_0^{2\pi} e^{-n\pi r^2} \mathbb{1}_{\text{first edge above } st} 2r \sin \frac{\alpha_1 - \alpha_2}{2} r^3 2A(\text{triangle}) d\alpha_1 d\alpha_2 dy_z dx_z dr \]

\[ = 4n^3 \left( \int_0^\infty e^{-n\pi r^2} r^5 dr \right) \cdot \left( \int_{h=\frac{\pi}{r}}^{-1} \int_0^{2\pi} \mathbb{1}_{\text{first edge above } st} \sin \frac{\alpha_1 - \alpha_2}{2} A(\text{triangle}) d\alpha_1 dh \right) \]
\[ \mathbb{E}[\text{length}] = \mathbb{E} \left[ \sum_{\text{triangle} \in X_n^3} \mathbb{1}_{[\text{triangle is Delaunay}]} \mathbb{1}_{[\text{first edge above } st]} \text{length(first edge)} \right] \]

\[ = n^3 \int_{(\mathbb{R}^2)^3} \mathbb{P}[\text{triangle is Delaunay}] \mathbb{1}_{[\text{first edge above } st]} \text{length(first edge)} \, dt_{\text{triangle}} \]

\[ = n^3 \int_{r=0}^{\infty} \int_{x=0}^{1} \int_{y_z=-r}^{r} \int_{[0,2\pi]^3} e^{-n\pi r^2} \mathbb{1}_{[\text{first edge above } st]} 2r \sin \frac{\alpha_1 - \alpha_2}{2} r^3 2A(\text{triangle}) \, d\alpha_1 \, dy_z \, dx_z \, dr \]

\[ = 4n^3 \left( \int_{r=0}^{\infty} e^{-n\pi r^2} r^5 \, dr \right) \cdot \left( \int_{h=\frac{y_z}{r}=-1}^{1} \int_{[0,2\pi]^3} \mathbb{1}_{[\text{first edge above } st]} \sin \frac{\alpha_1 - \alpha_2}{2} A(\text{triangle}) \, d\alpha_1 \, dh \right) \]

\[ = 4n^3 \cdot \frac{1}{\pi^3 n^3} \cdot \frac{35\pi}{12} = \frac{35}{3\pi^2} \]
# Walking in Delaunay triangulations

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<tr>
<th>Path Type</th>
<th>Expected Length (Experiments)</th>
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<td></td>
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<td>$1.16$</td>
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<tr>
<td>Voronoi path</td>
<td></td>
<td>[Baccelli et al., 2000]</td>
</tr>
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Expected length (experiments) theory

$\geq 1 + 10^{-11}$

Numerical integration

$\frac{35}{3\pi^2} \approx 1.18$

$\frac{4}{\pi} \approx 1.27$

[Baccelli et al., 2000]
Bad edge = almost horizontal edge
Many bad edges ⇐ length close to 1
\( \mathbb{P}[\text{bad}] = \text{small constant} \)
difficult dependencies to handle
Make a grid
Make a grid
Make a grid

If $\mathbb{E}[\# \in cell] = \text{constant}$

$\#$ possible paths $= 4^n$ (too big)
Make a grid

If $\mathbb{E}[\# \in \text{cell}] = \text{constant}$

$\# \text{ possible paths} = 4^n$ (too big)

big cells $\sqrt{n} \times \sqrt{n}$  $4\sqrt{n} \times n$
A good cell?

No Delaunay circles go outside
A good cell?

No Delaunay circles go outside

No short path from left to right
A good cell?

No Delaunay circles go outside

No short path from left to right

Not enough edges to make a short path
A good cell?

No Delaunay circles go outside

No short path from left to right

Not enough edges to make a short path

Choose ♯ points in cell,

Choose what ”short path” means

\[ P \left[ \text{length} \geq 1 + 2.5 \times 10^{-11} \right] \leq O \left( \frac{1}{\sqrt{n}} \right) \]
Walking in Delaunay triangulations

Voronoi path

\[ \frac{4}{\pi} \approx 1.27 \]

[Baccelli et al., 2000]

Voronoi path in higher dimension
Walking in Delaunay triangulations

Voronoi path

\[ \frac{4}{\pi} \approx 1.27 \]

Voronoi path in higher dimension

\[ \mathbb{E} [\ell(VP_X)] = \frac{1}{2} \int_{-\infty}^{\infty} \int_0^\infty \int_{(S_{d-1})^2} \mathbb{P} [B((x,0,...,0),r) \cap (S_{d-1})^2] \]

\[ \cdot r \| u_1 u_2 \| \det(J_\Phi) \, d\alpha_1,1 \ldots d\alpha_{1,d-1} d\alpha_2,1 \ldots \]

Integral form
Walking in Delaunay triangulations

Voronoi path

Voronoi path in higher dimension

\[ E[\ell(VPX)] = \frac{1}{2} \int_{-\infty}^{\infty} \int_{0}^{\infty} \int_{(S_{d-1})^2} \mathbb{P}[B((x,0,...,0),r) \cap (S_{d-1})^2] \cdot r \|u_1u_2\| \det(J_\Phi) d\alpha_1,1 \ldots d\alpha_1,d-1 d\alpha_2,1 \ldots \]

Integral form

Use Taylor expansion to be able to integrate
Walking in Delaunay triangulations

Voronoi path

1.27

$\frac{4}{\pi} \approx 1.27$

[Baccelli et al., 2000]

Voronoi path in higher dimension

$\Gamma \left( \frac{d}{2} \right)^4 2^{4d-5} d \pi^2 (2d - 2)! \left( 1 - \frac{d - 1}{4d^2 - 1} \right) \sqrt{2} \leq \mathbb{E} [\ell(VP_X)] \leq \frac{\Gamma}{\pi}$
Walking in Delaunay triangulations

\[ 4 \pi \simeq 1.27 \]

[Baccelli et al., 2000]

Voronoi path

Voronoi path in higher dimension

Asymptotic behavior between

\[ \sqrt{\frac{2d}{\pi}} - \frac{1}{4\sqrt{2d\pi}} + O(d^{3/2}) \]

\[ + \frac{3}{4\sqrt{2d\pi}} + O(d^{3/2}) \]
Walking in Delaunay triangulations

Voronoi path 1.27 \[ \frac{4}{\pi} \approx 1.27 \]

[Baccelli et al., 2000]

Voronoi path in higher dimension

asymptotic behavior

\[ \sqrt{\frac{2d}{\pi}} \]
Walking in Delaunay triangulations

Voronoi path

\( \frac{4}{\pi} \approx 1.27 \) [Baccelli et al., 2000]

Voronoi path in higher dimension

<table>
<thead>
<tr>
<th>( d )</th>
<th>( k )</th>
<th>lower bound</th>
<th>( \sim )</th>
<th>exact value</th>
<th>upper bound</th>
<th>( \sim )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>41</td>
<td>( \frac{788984278470257640690697143}{745000536337515228912680960} \sqrt{2} )</td>
<td>1.49770</td>
<td>1.500</td>
<td>( \frac{4523370364712510658076963509}{4264485828690604413776035840} \sqrt{2} )</td>
<td>1.50007</td>
</tr>
<tr>
<td>4</td>
<td>7</td>
<td>( \frac{102494570}{8729721} \sqrt{2} )</td>
<td>1.6823</td>
<td>1.698</td>
<td>( \frac{121774997}{10270260} \sqrt{2} )</td>
<td>1.6990</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>( \frac{135}{104} \sqrt{2} )</td>
<td>1.8357</td>
<td>1.875</td>
<td>( \frac{21305}{16016} \sqrt{2} )</td>
<td>1.8812</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>( \frac{3014656}{225225} \sqrt{2} )</td>
<td>1.9179</td>
<td>2.04</td>
<td>( \frac{753664}{51975} \sqrt{2} )</td>
<td>2.0778</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>( \frac{210}{143} \sqrt{2} )</td>
<td>2.0768</td>
<td>2.2</td>
<td>( \frac{225}{143} \sqrt{2} )</td>
<td>2.2252</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>( \frac{2080374784}{134008875} \sqrt{2} )</td>
<td>2.2244</td>
<td>2.3</td>
<td>( \frac{130023424}{7882875} \sqrt{2} )</td>
<td>2.3635</td>
</tr>
</tbody>
</table>

Numerical integration
Thank you