Walking in Poisson Delaunay triangulations

Olivier Devillers

[D. & Hemsley, *JoCG:7(1)*]

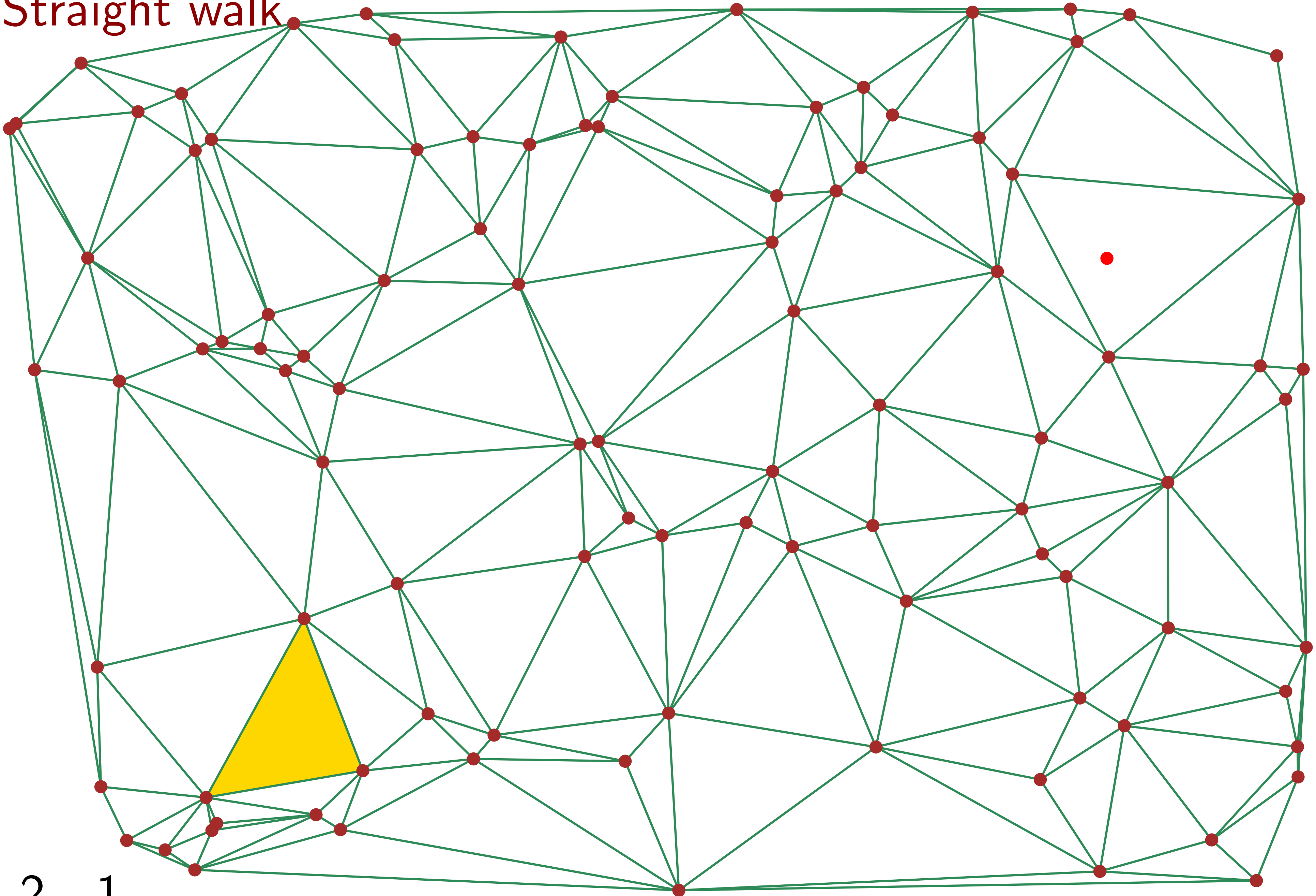
[Chenavier & D., *Hal*]

[de Castro & D., *Hal, DCG*]

[D. & Noizet, *Hal*]

Walking in Delaunay triangulations

Straight walk



Walking in Delaunay triangulations

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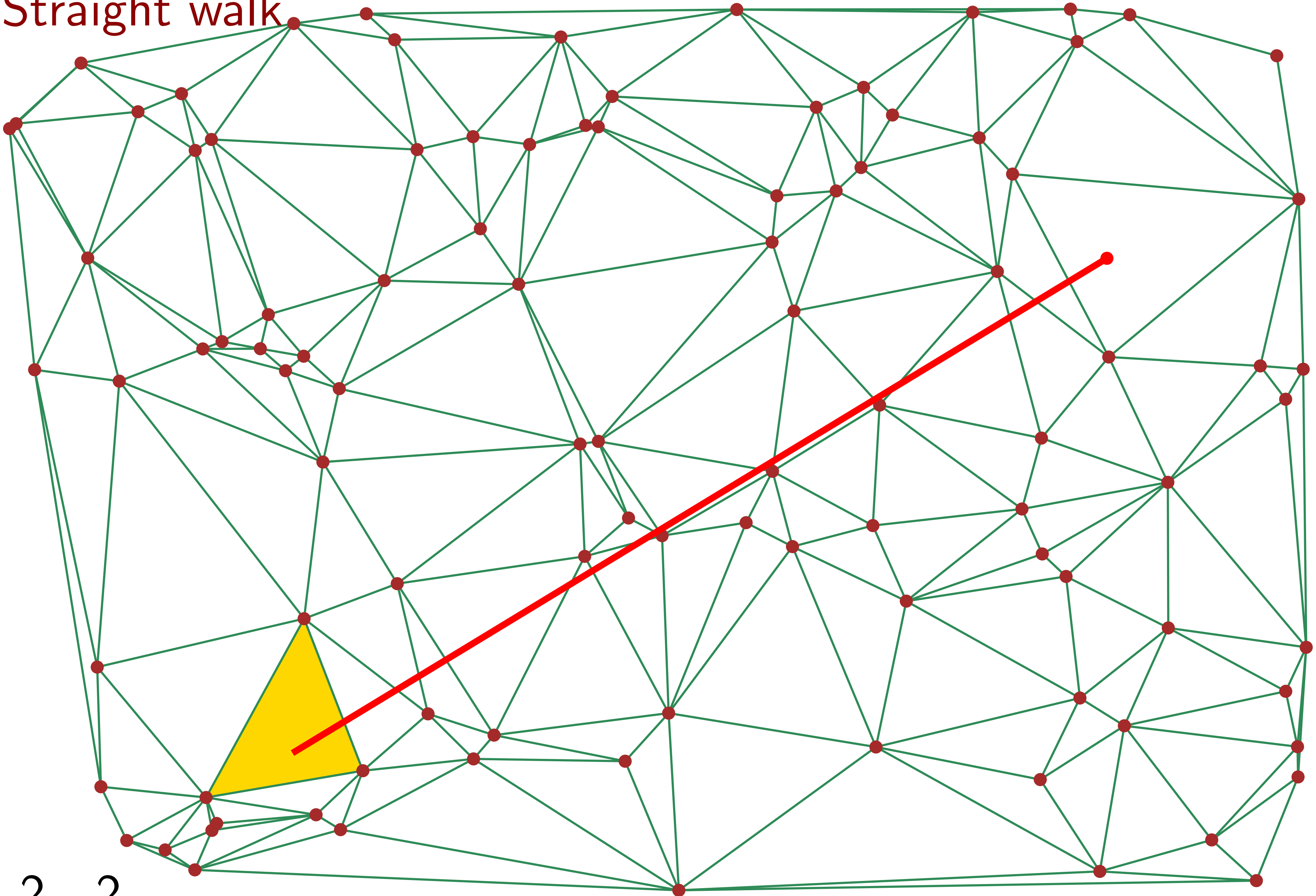

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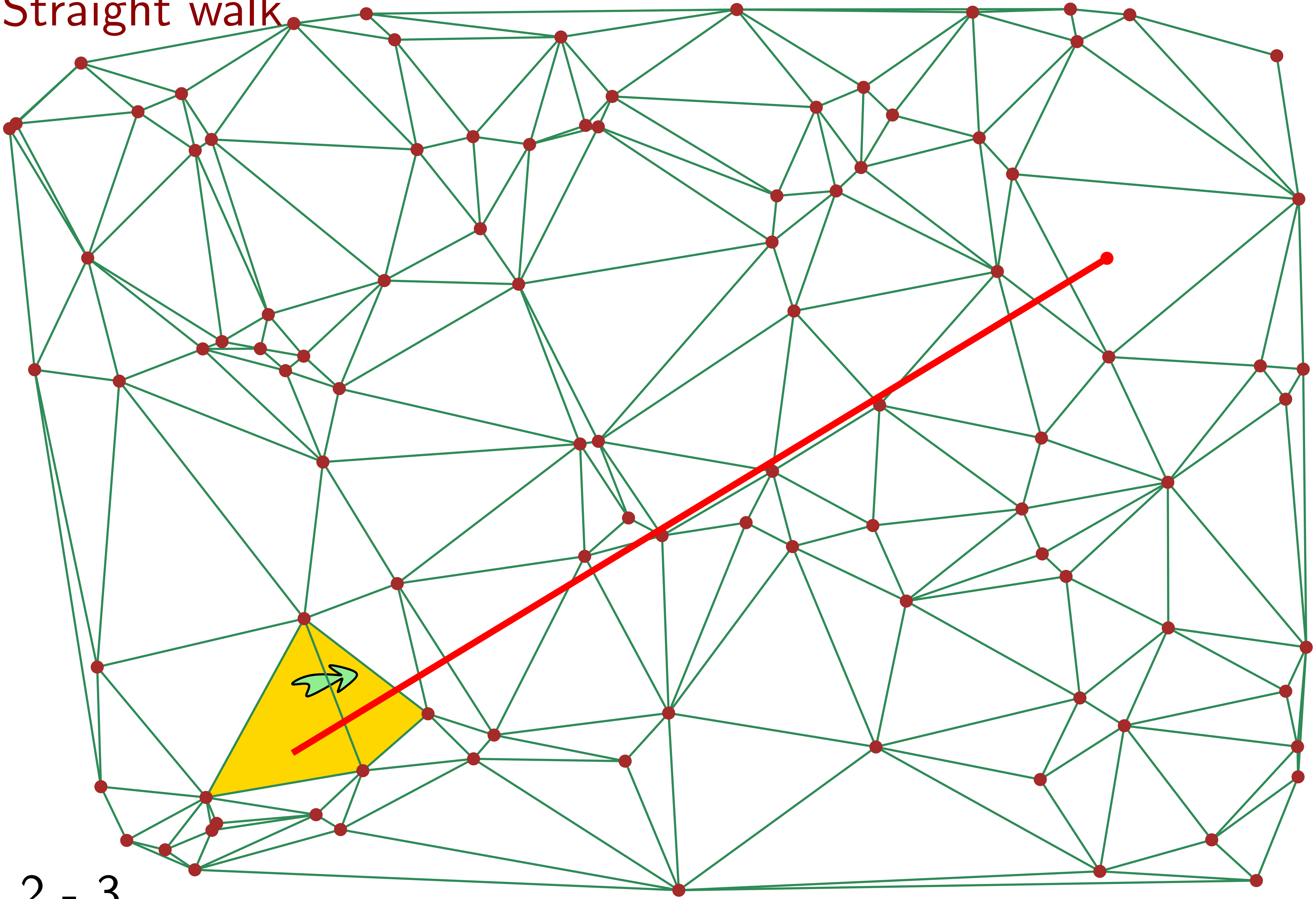

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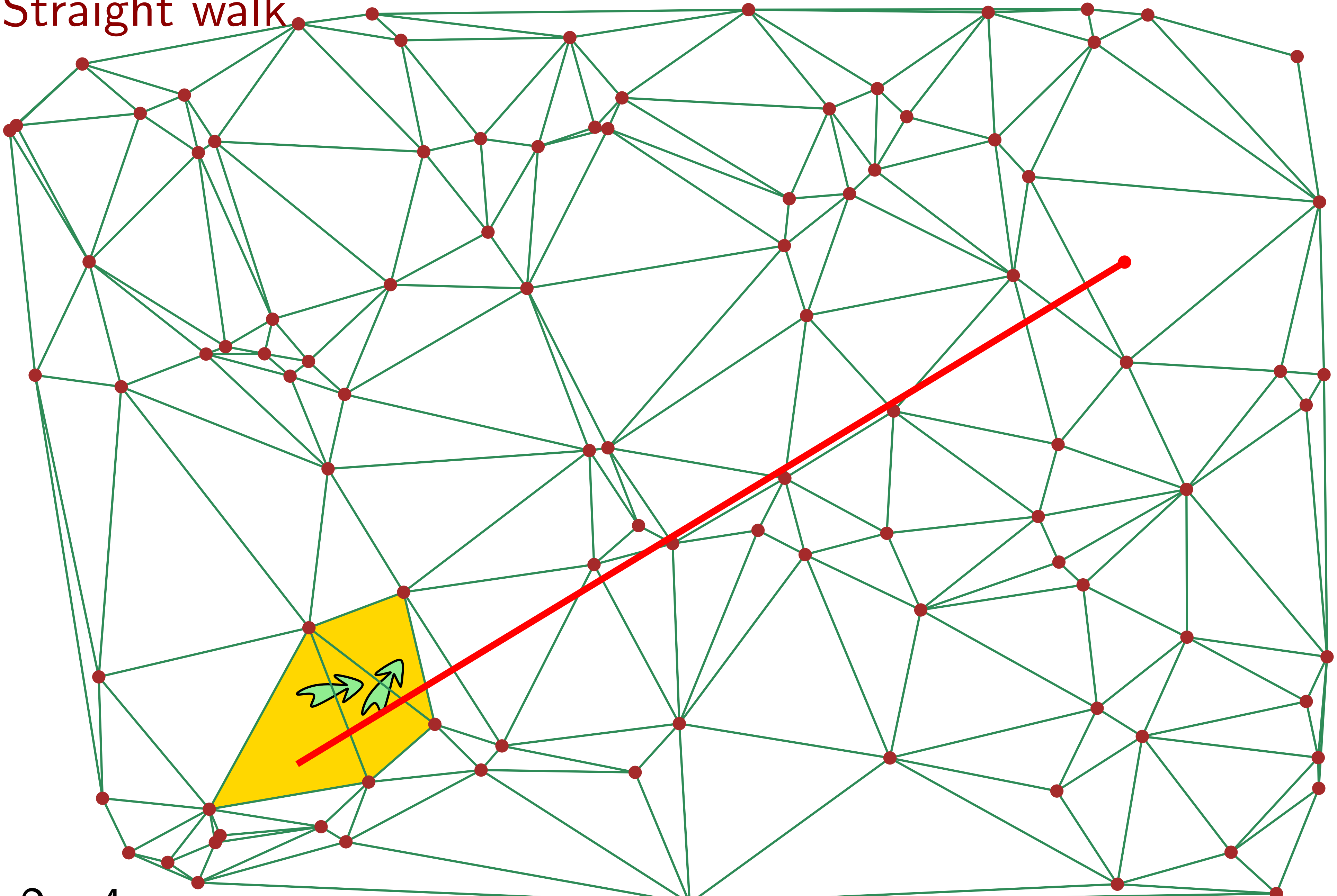

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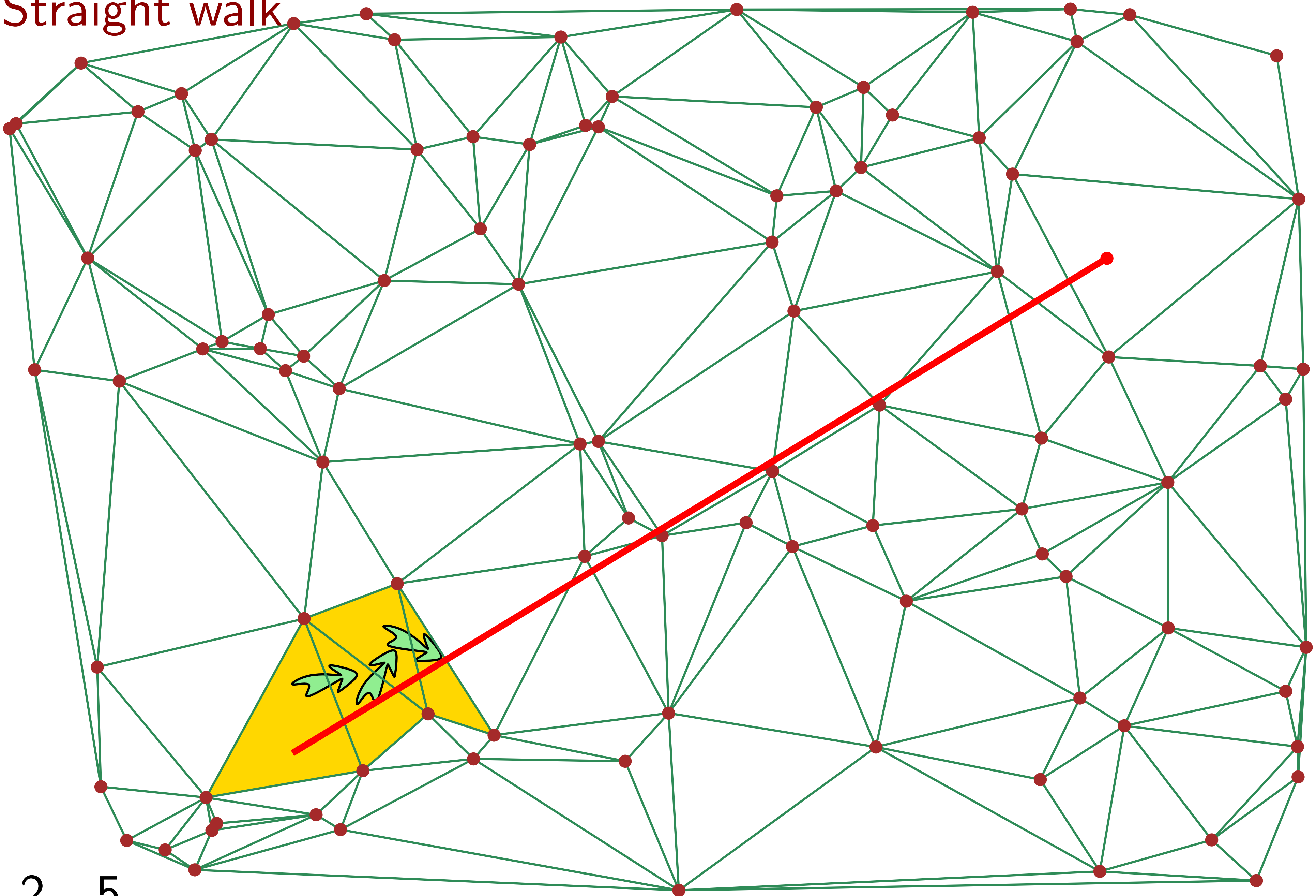

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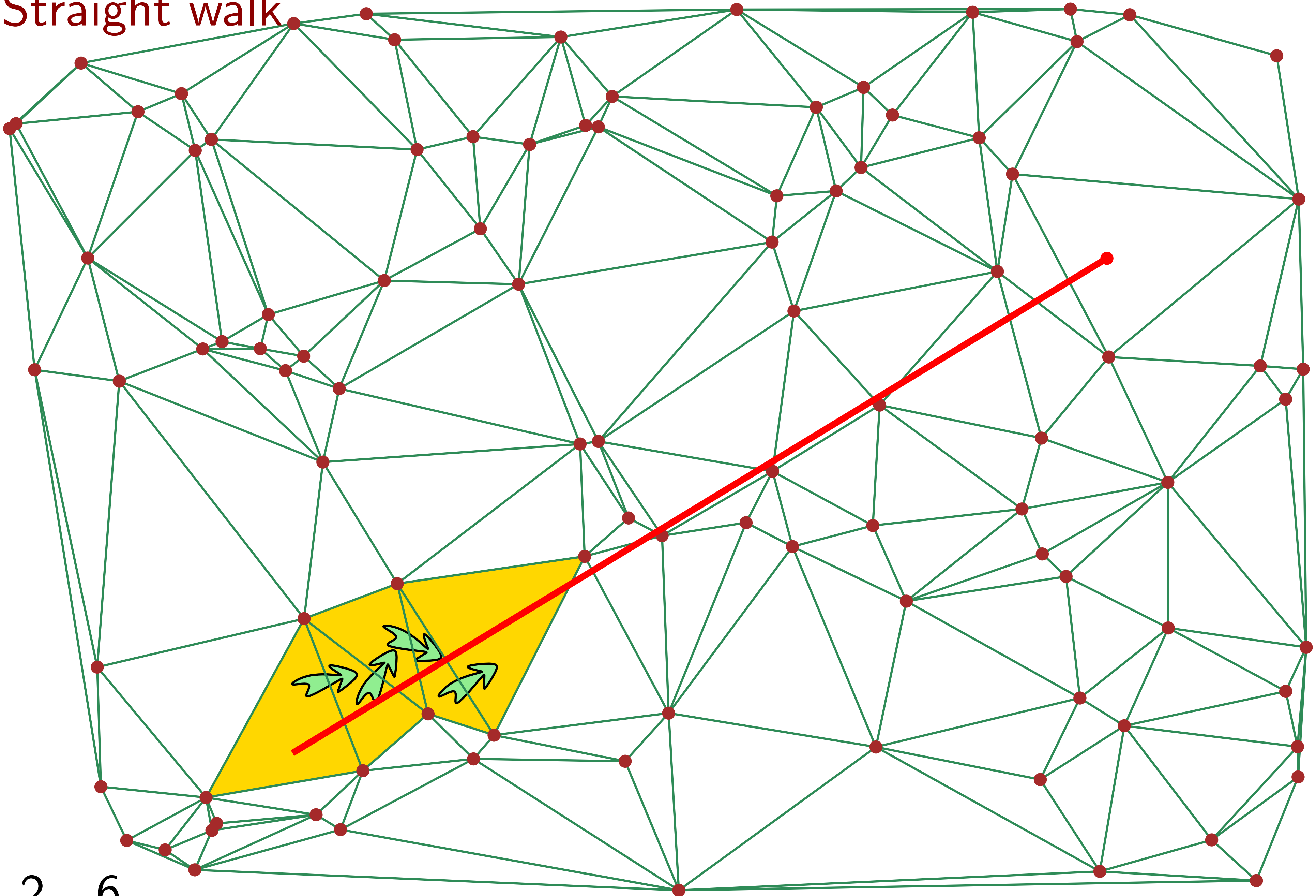

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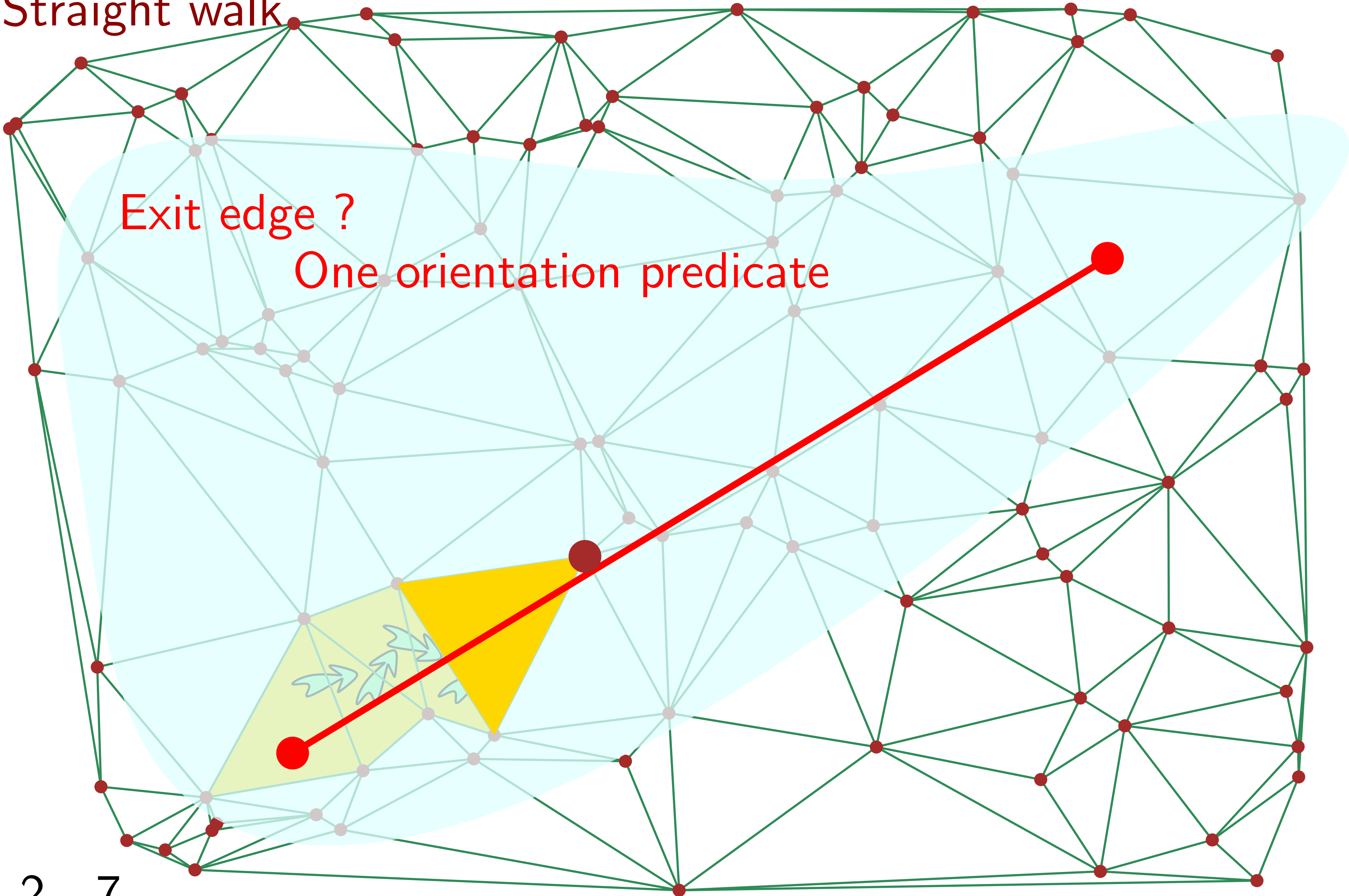

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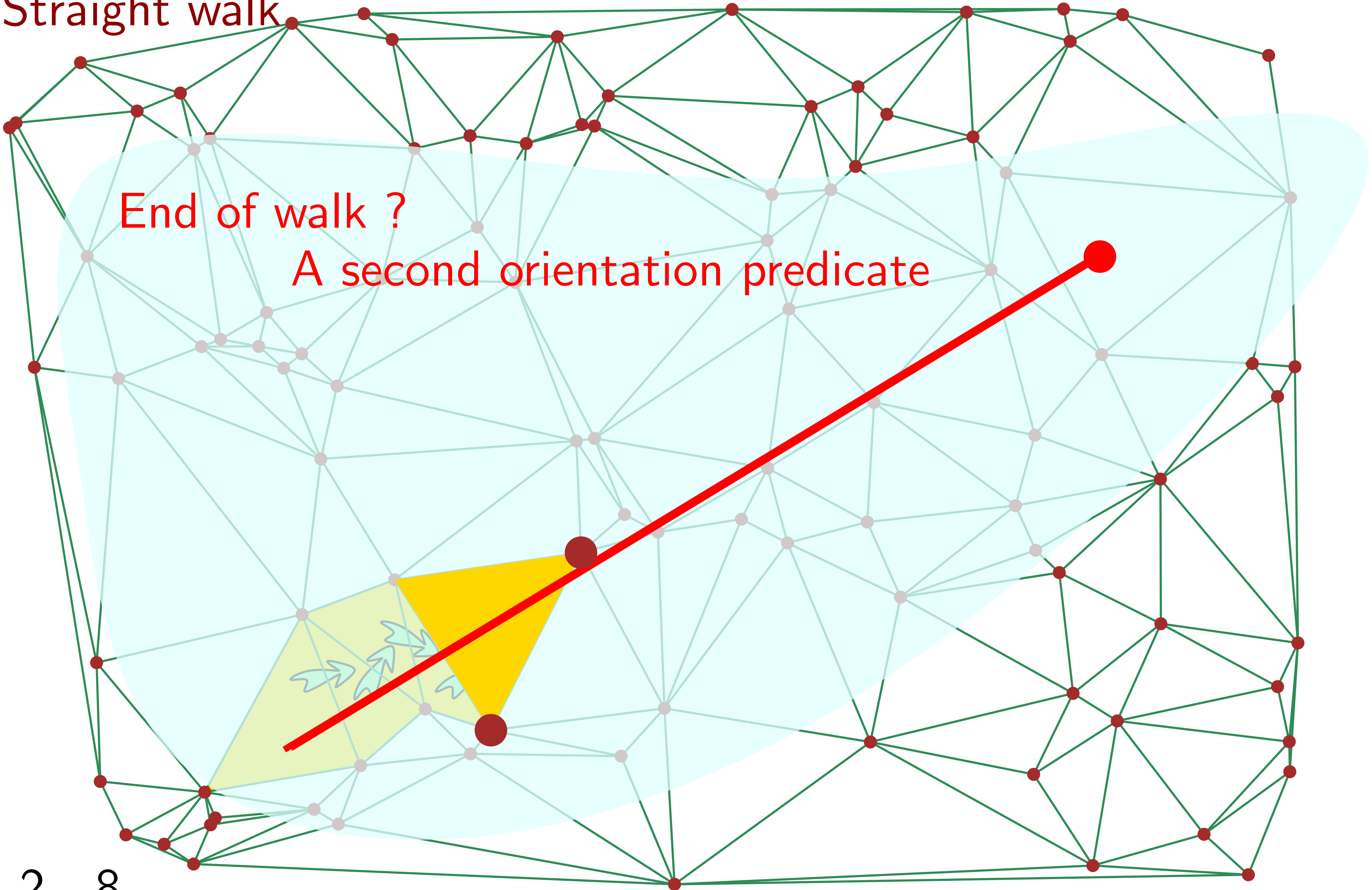

Walking in Delaunay triangulations

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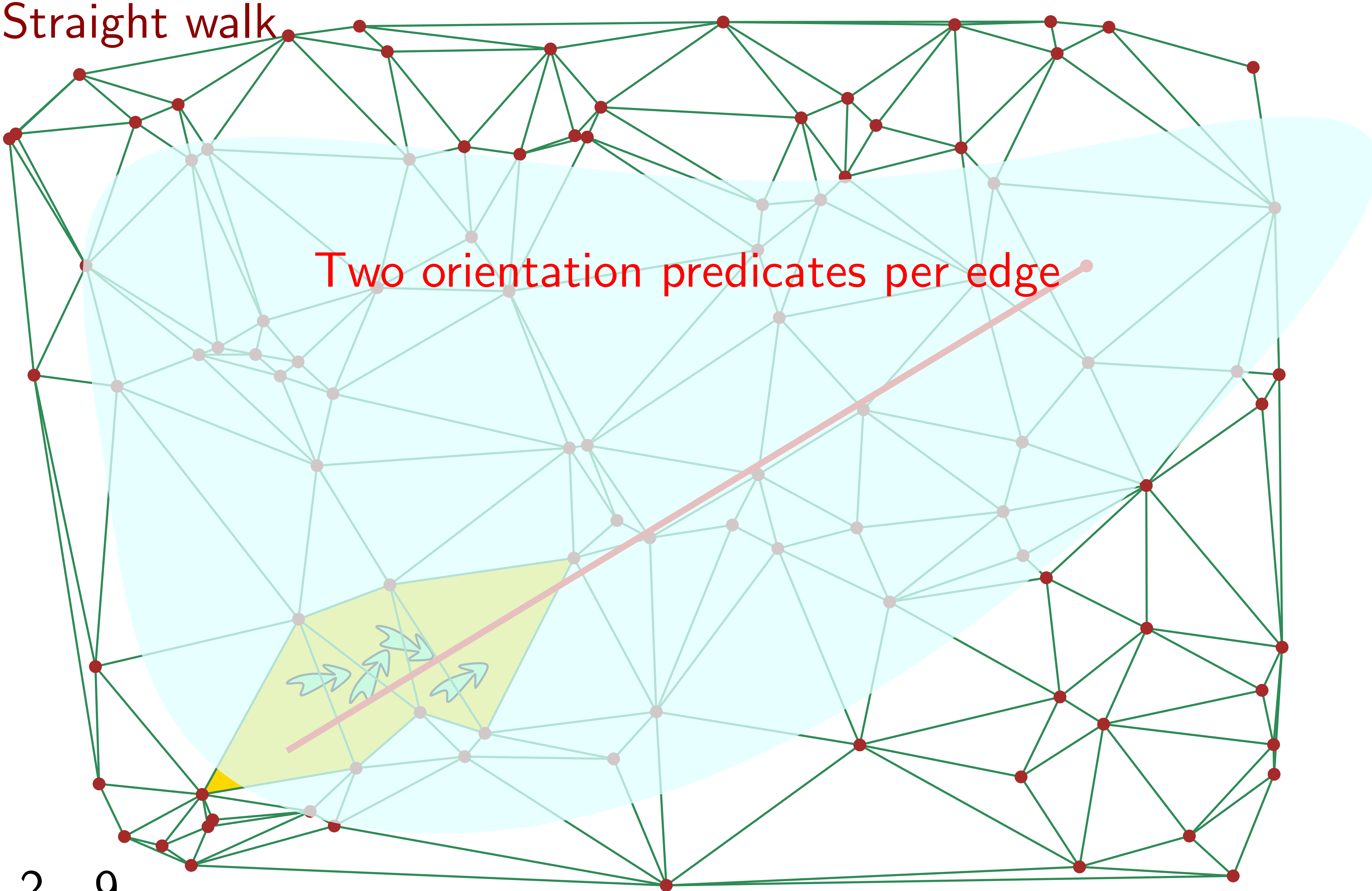
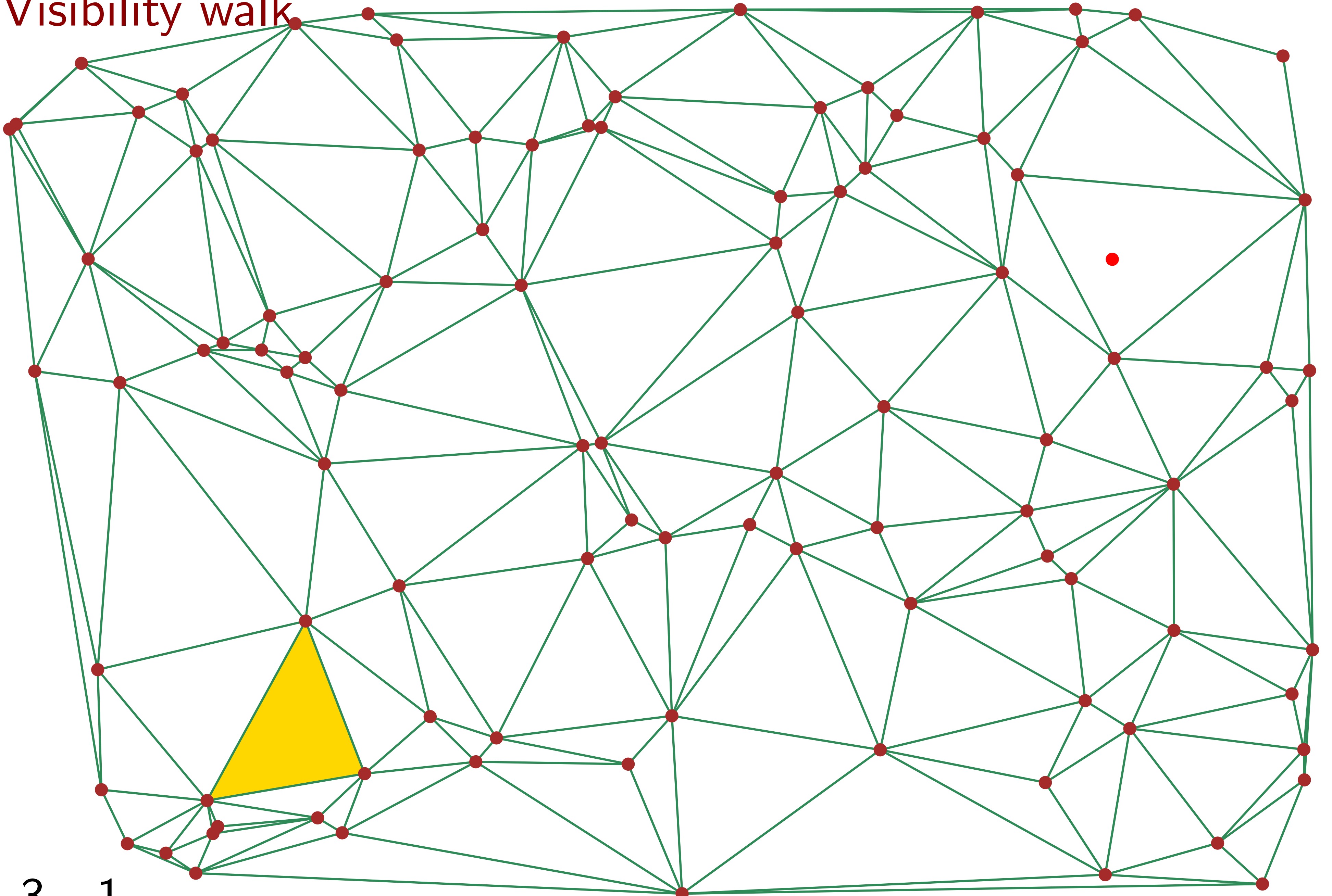


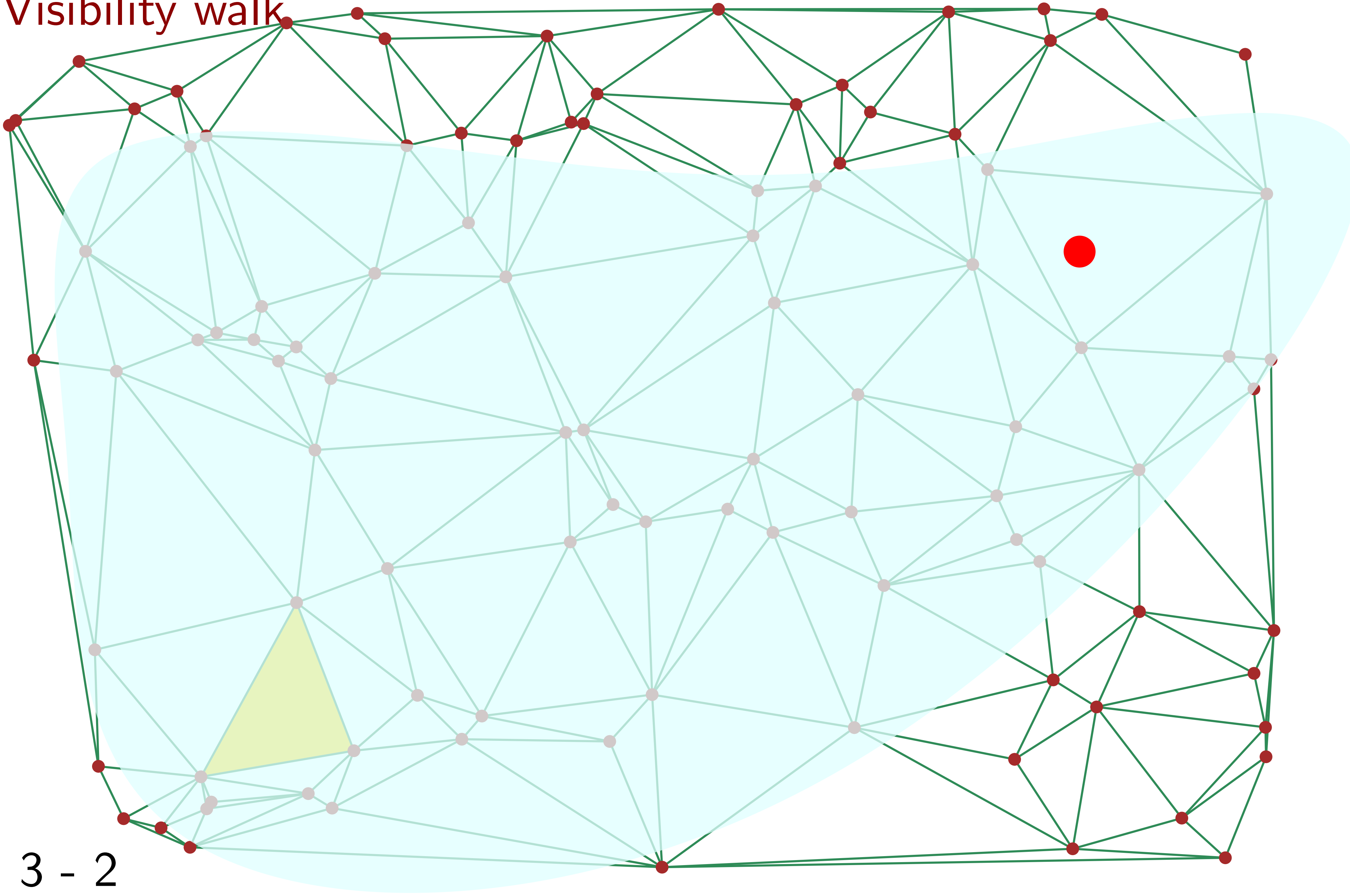
Walking in Delaunay triangulations

Visibility walk



Walking in Delaunay triangulations

Visibility walk



Walking in Delaunay triangulations

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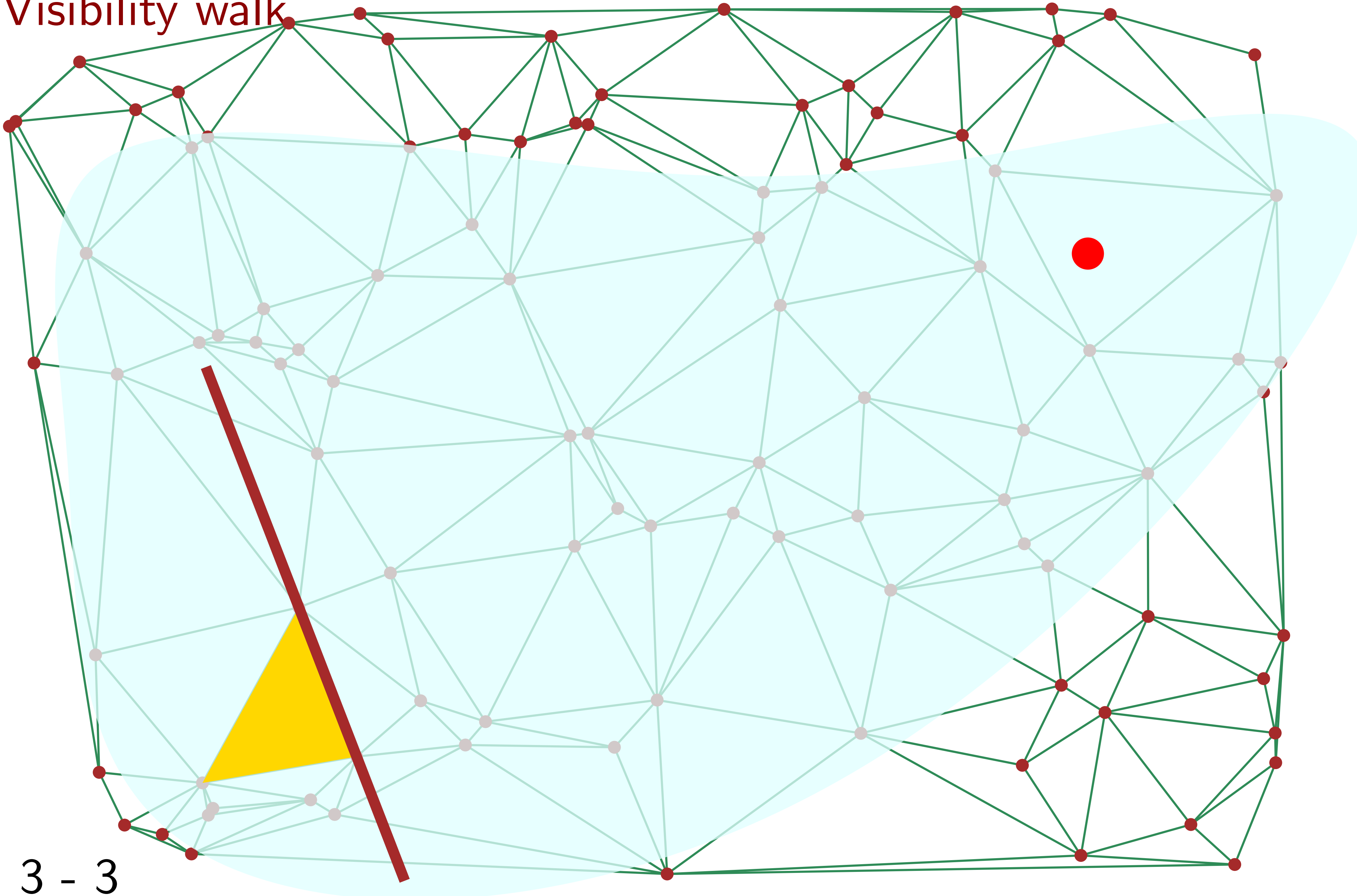
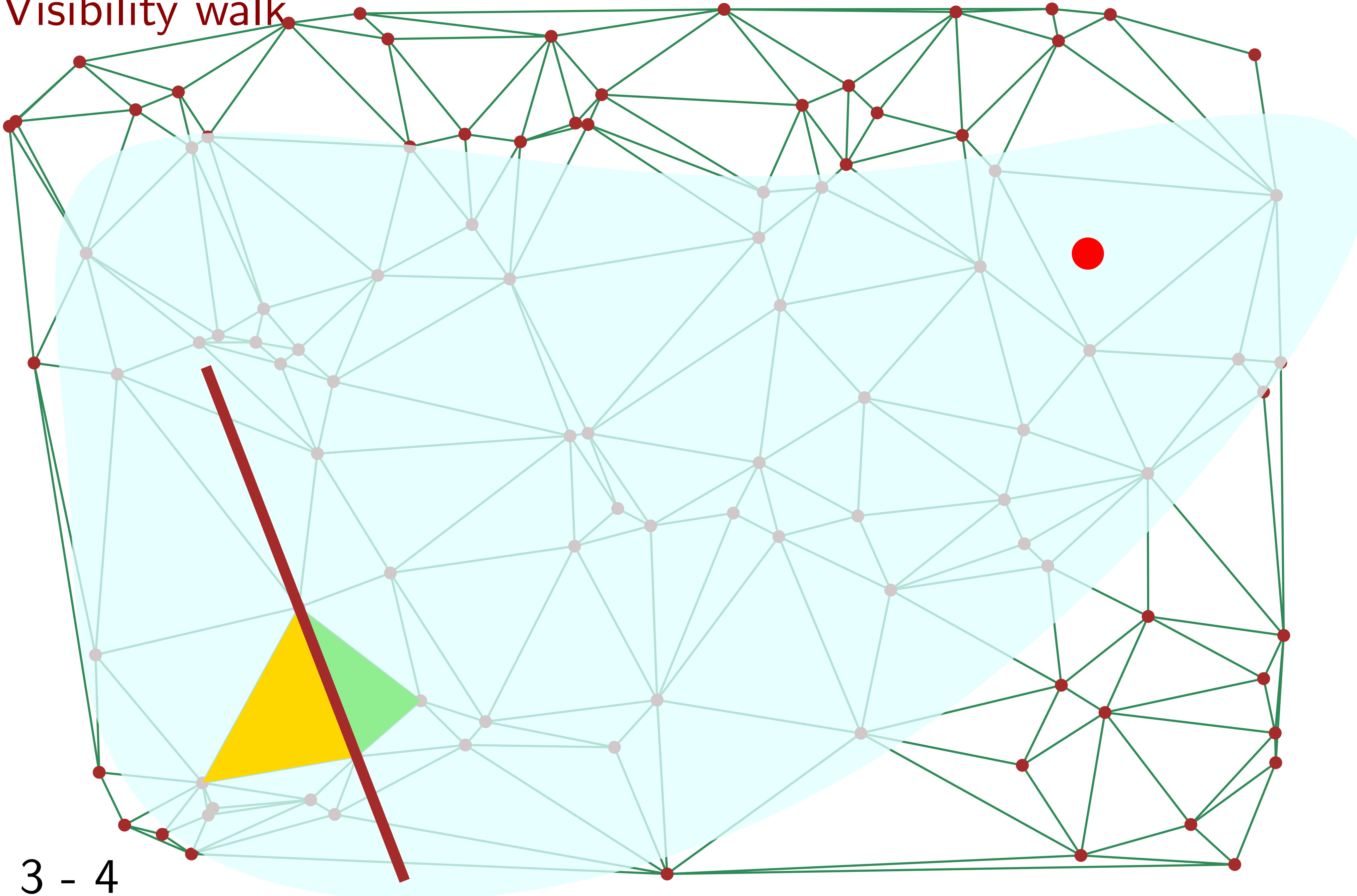


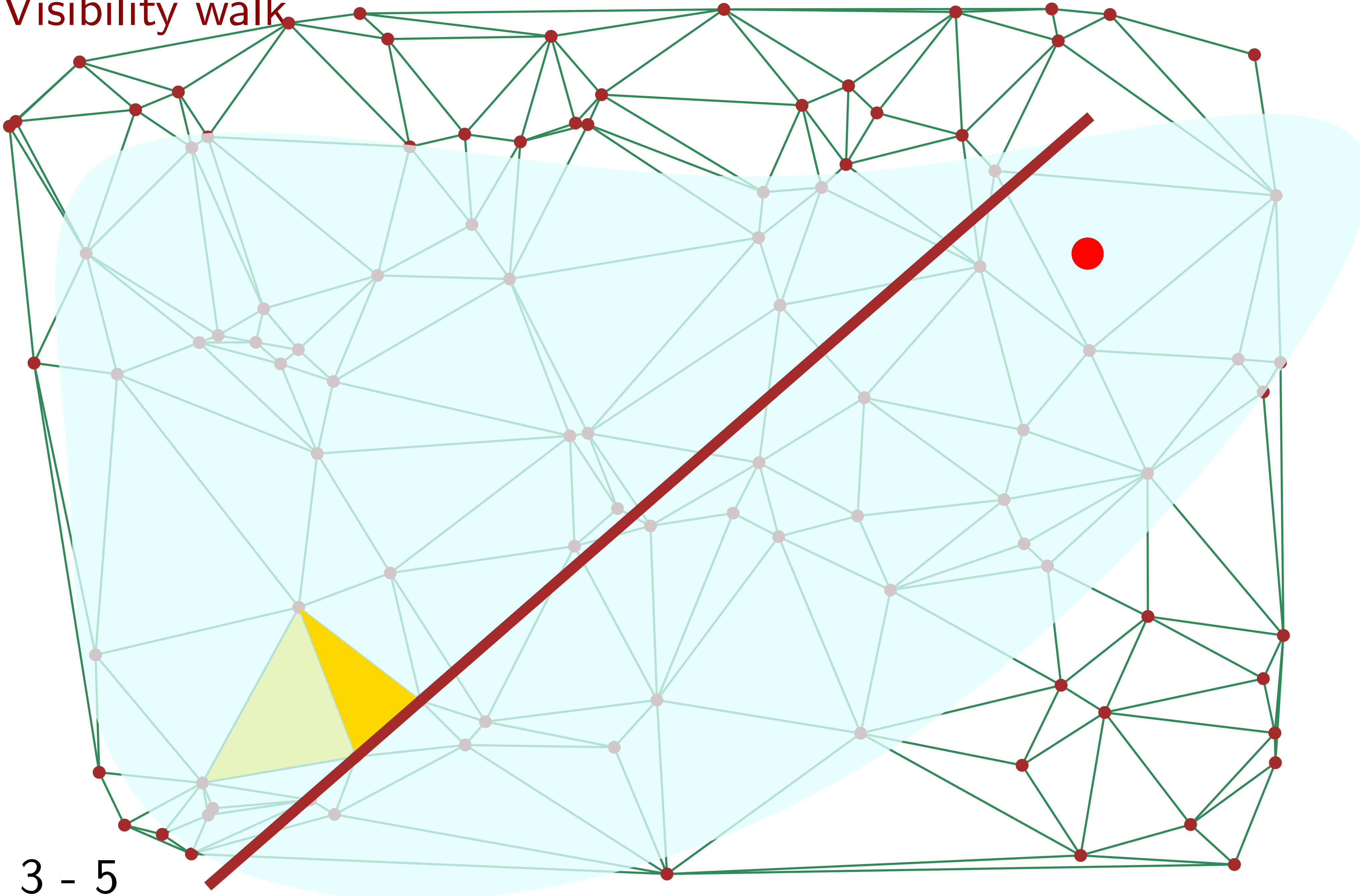
Walking in Delaunay triangulations

Visibility walk



Walking in Delaunay triangulations

Visibility walk



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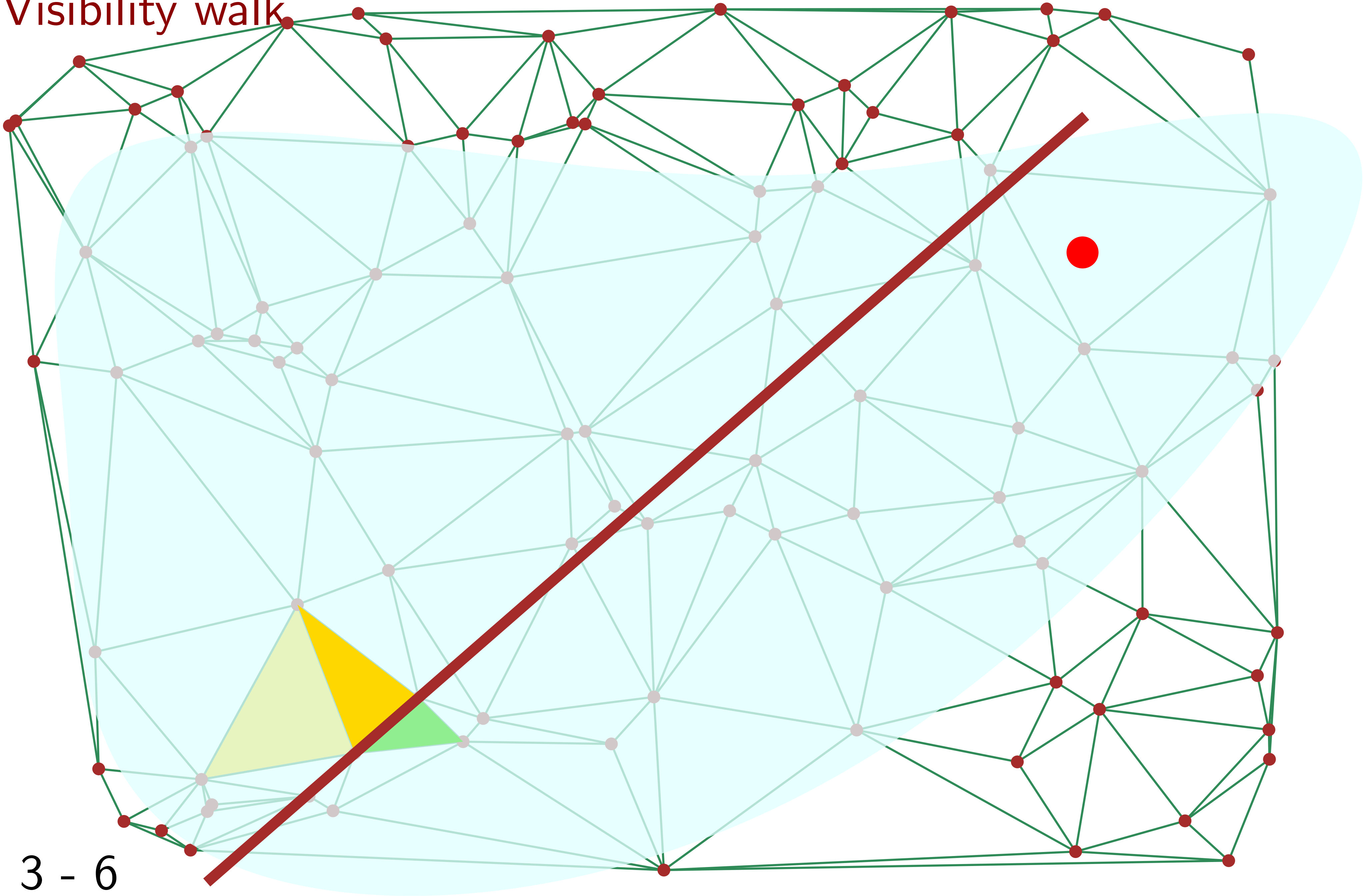

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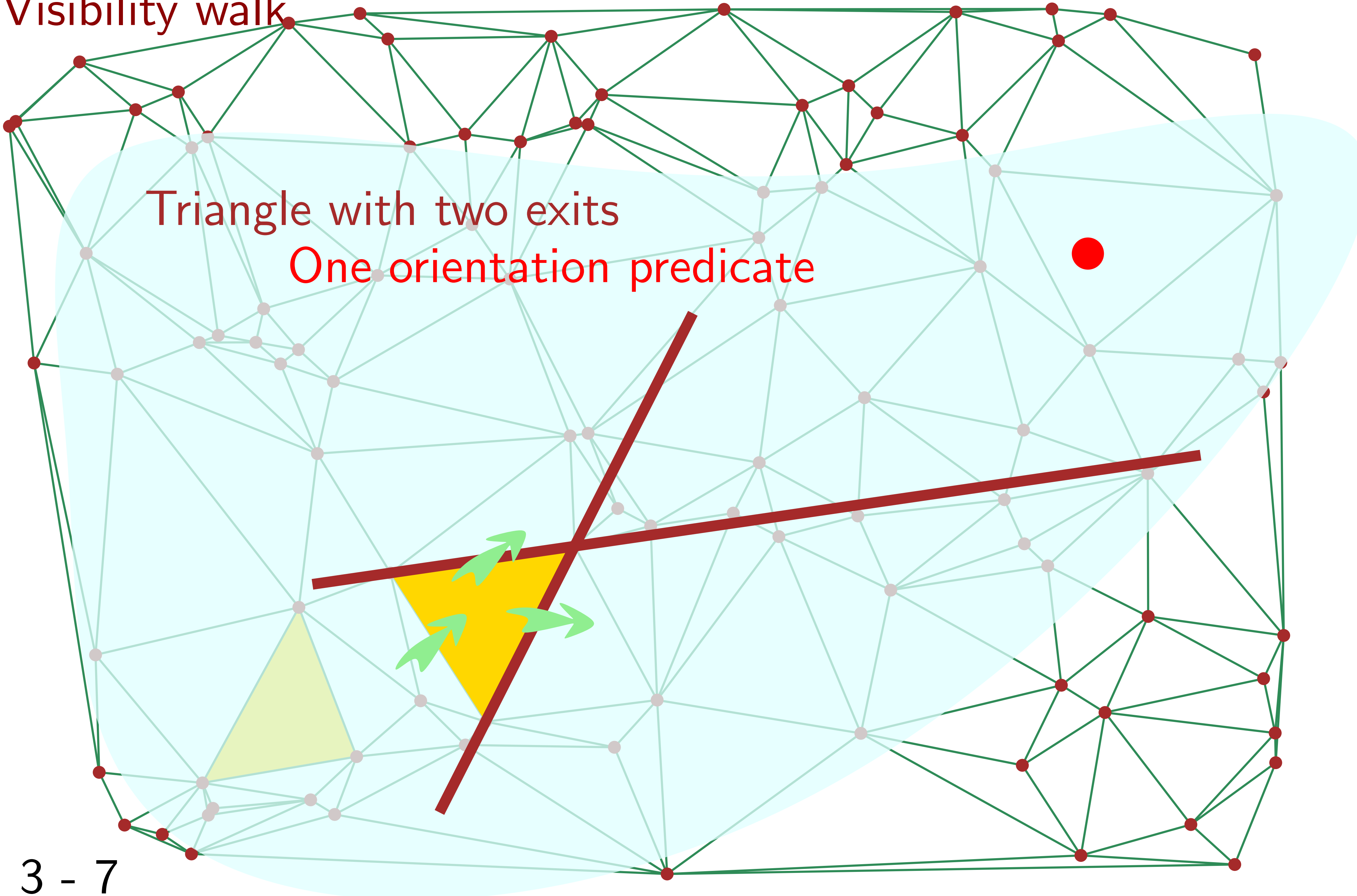

Walking in Delaunay triangulations

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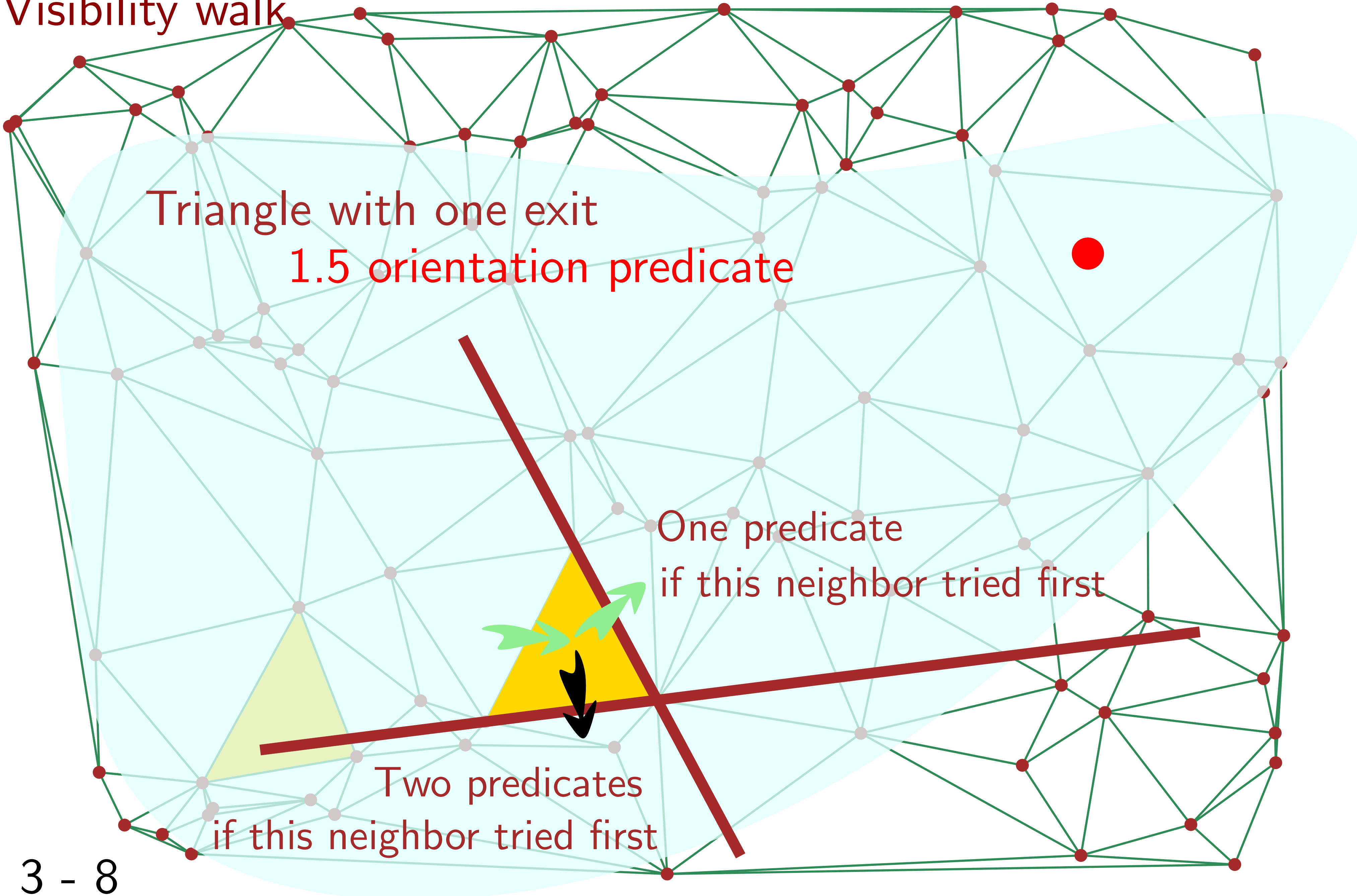

Walking in Delaunay triangulations

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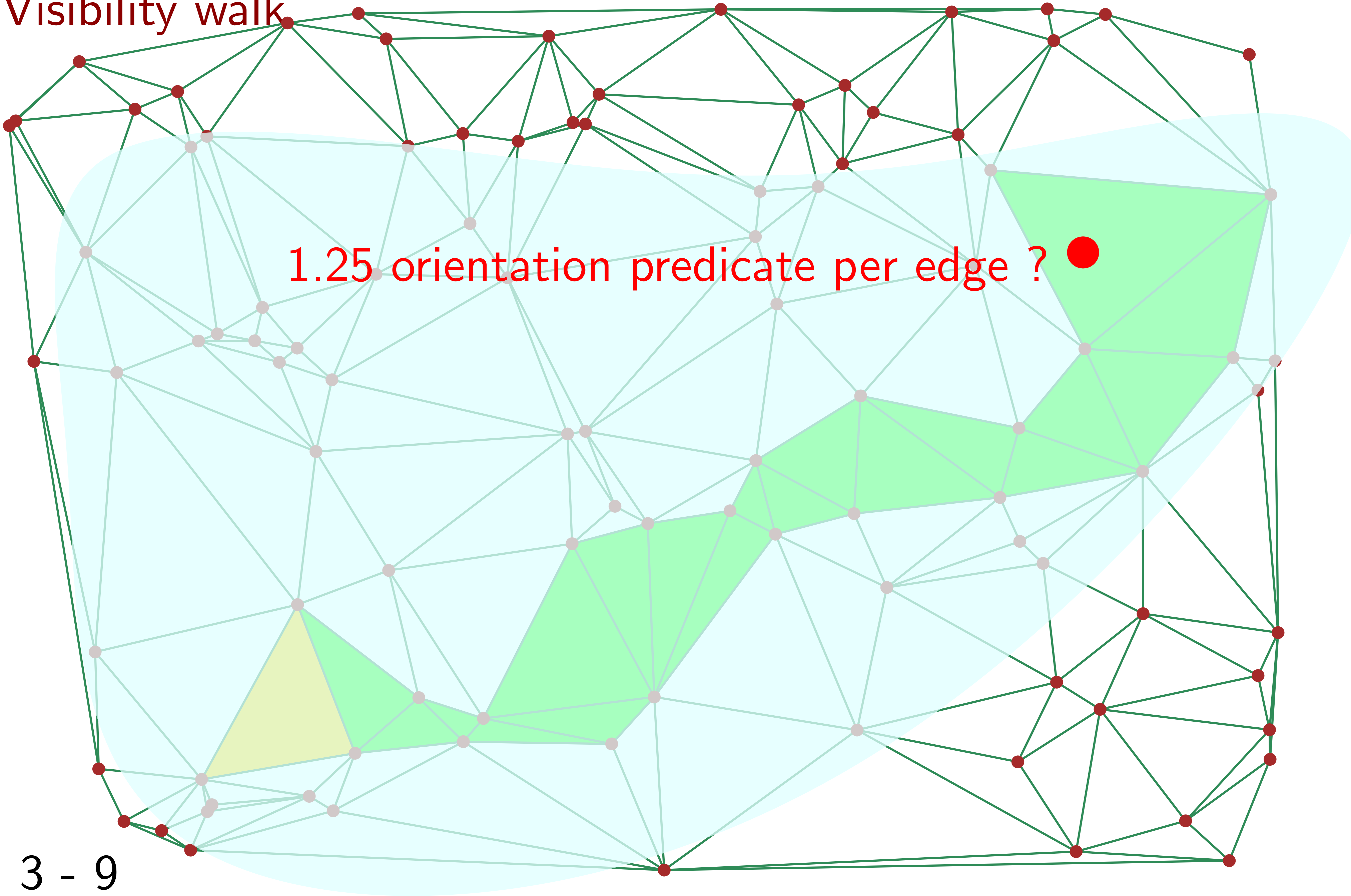


Walking in Delaunay triangulations

How many edges crossed ?

Straight walk

Visibility walk

Walking in Delaunay triangulations

How many edges crossed ?

Straight walk

$2n$

Visibility walk

∞ $\geq 2^{\sqrt[3]{n}}$ randomized

Worst case in a triangulation

Walking in Delaunay triangulations

How many edges crossed ?

Straight walk

$2n$

Visibility walk

$2n$

Worst case in a Delaunay triangulation

Walking in Delaunay triangulations

How many edges crossed ?

Straight walk

$2n$ $O(\sqrt{n})$ [Devroye, Lemaire, & Moreau, 2004]

$\frac{64}{3\pi^2}\sqrt{n} + O(\frac{1}{\sqrt{n}}) \simeq 2.16\sqrt{n}$

Visibility walk

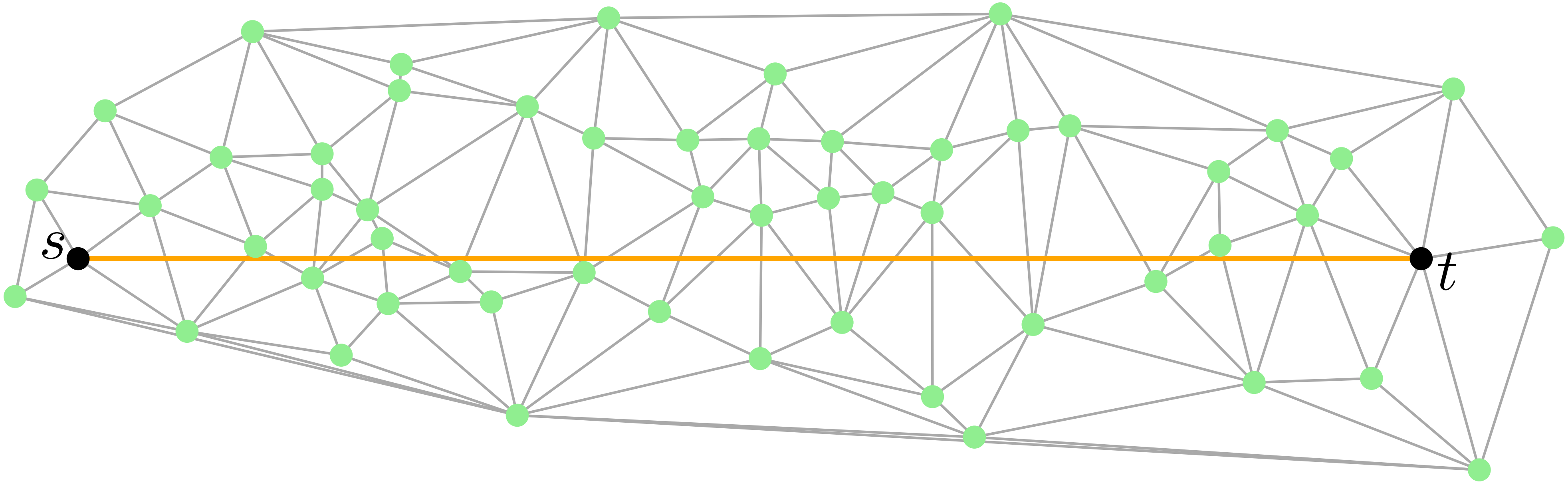
$2n$ $O(\sqrt{n})$

random

Worst case in a Delaunay triangulation

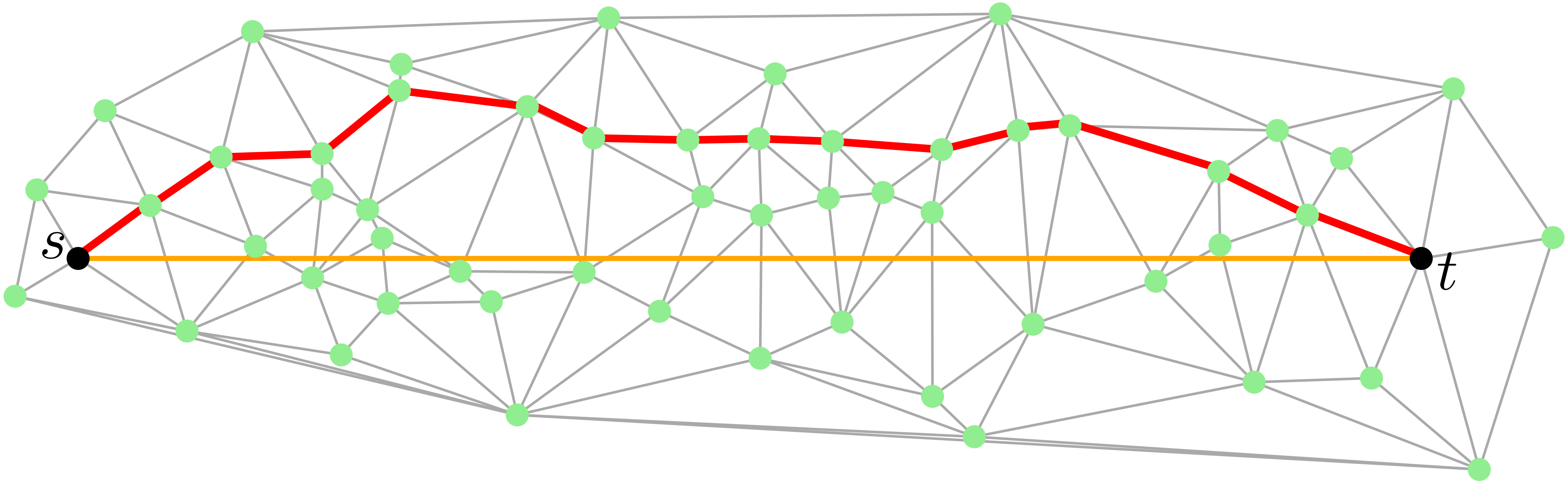
Walking in Delaunay triangulations

Walk between vertices



Walking in Delaunay triangulations

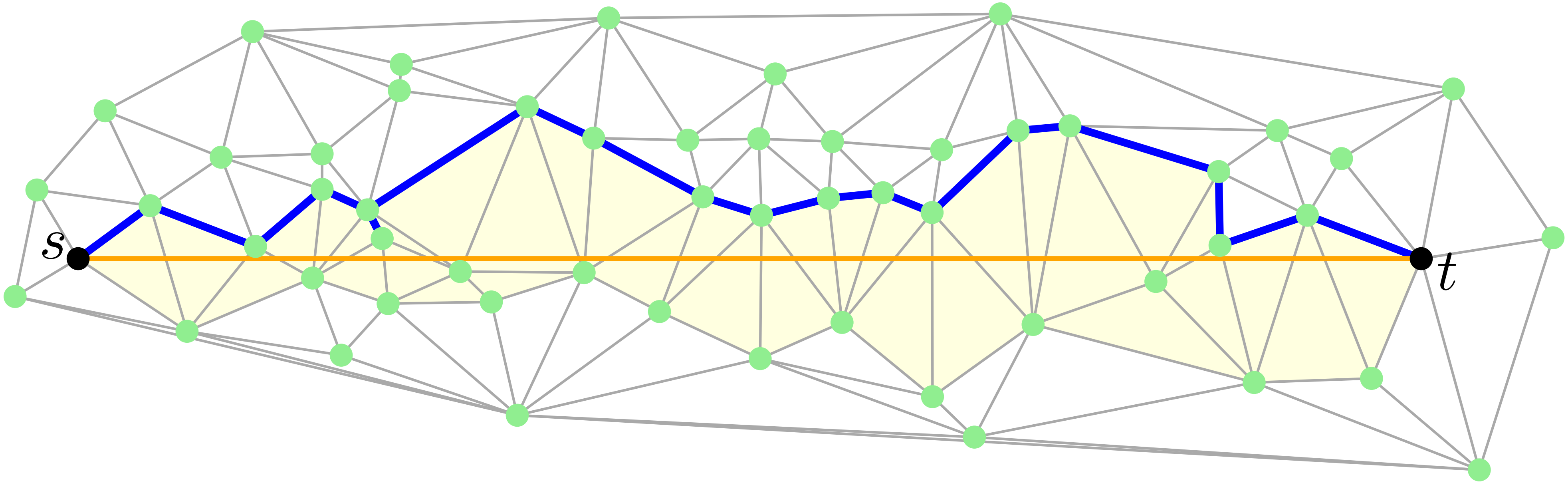
Walk between vertices



Shortest path

Walking in Delaunay triangulations

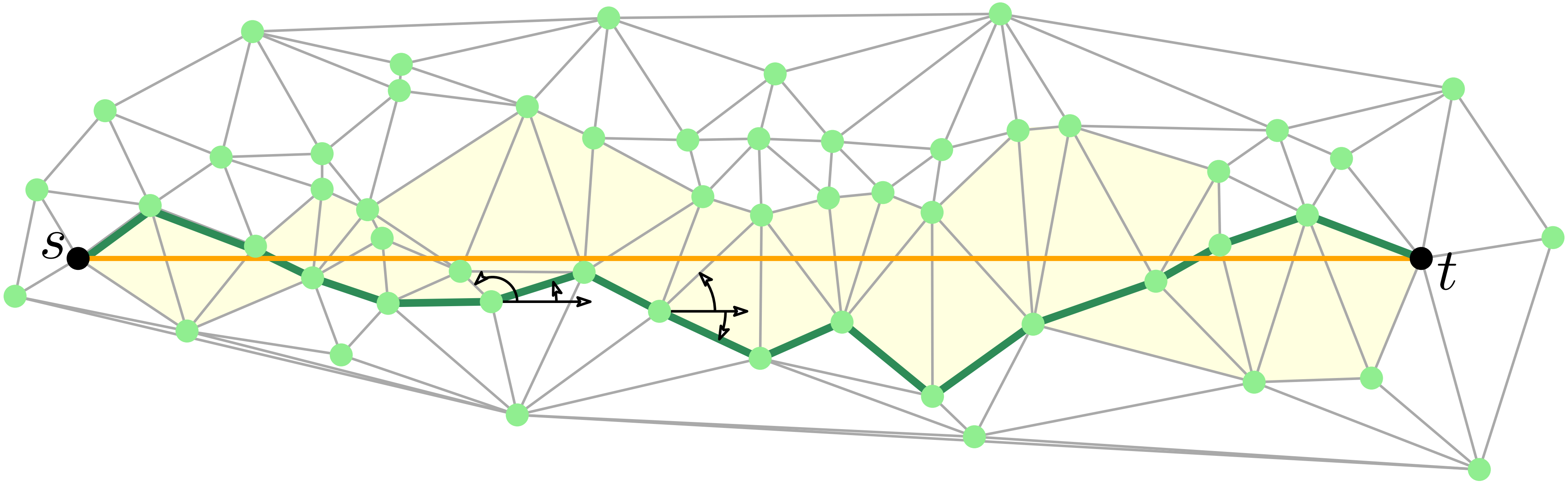
Walk between vertices



Upper path

Walking in Delaunay triangulations

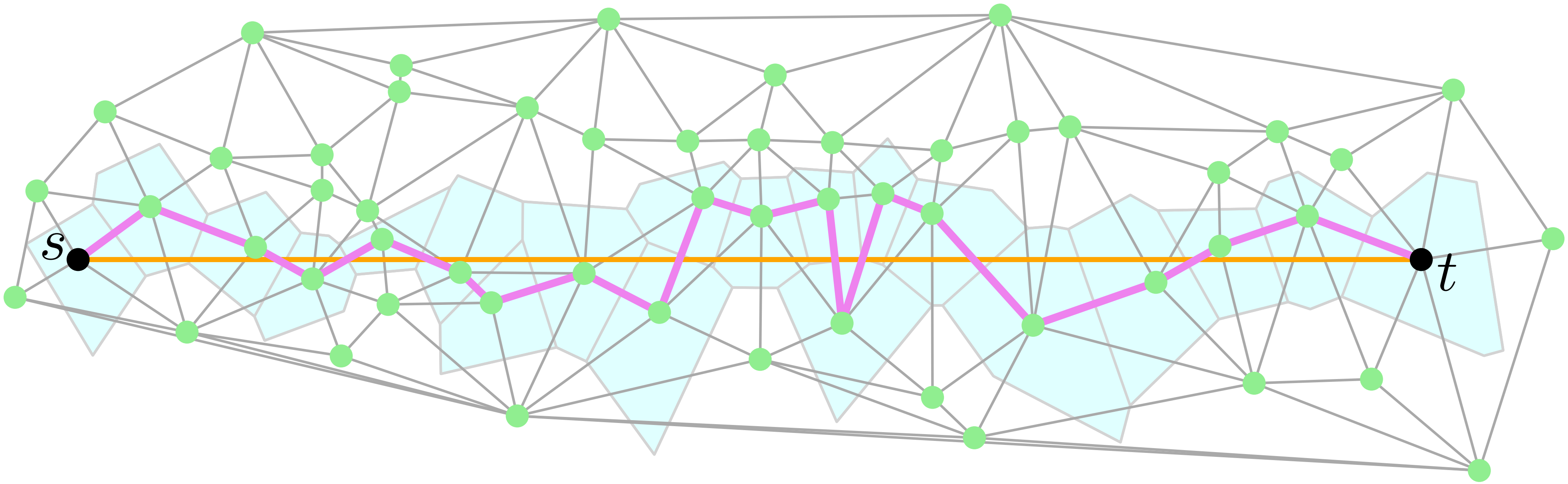
Walk between vertices



Compass walk

Walking in Delaunay triangulations

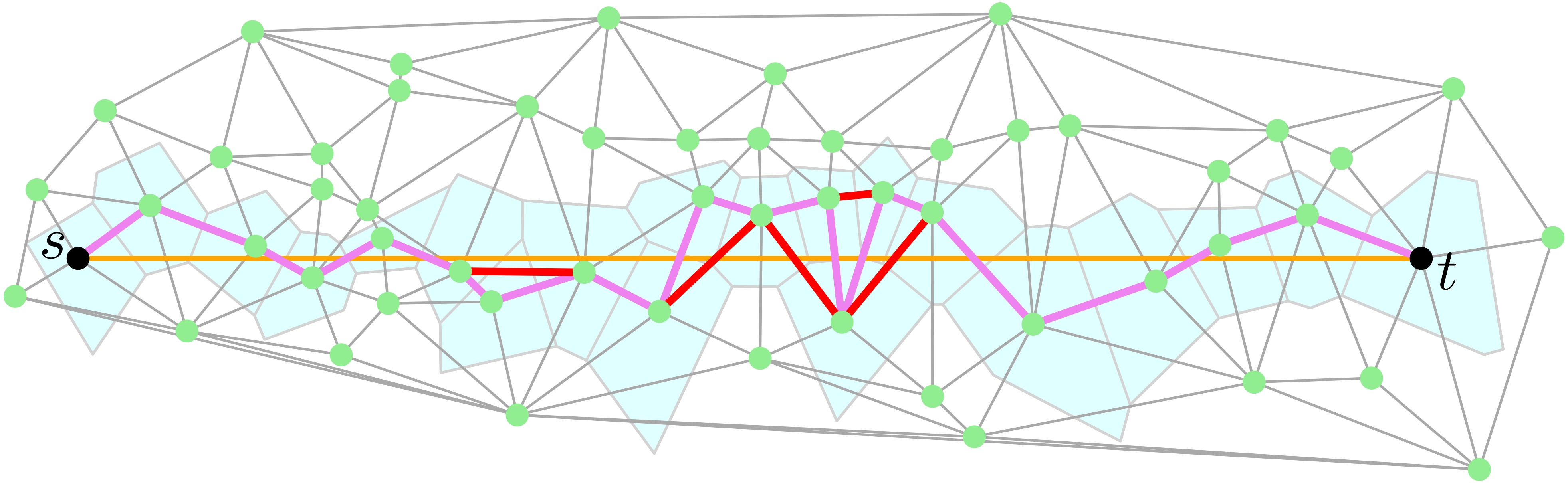
Walk between vertices



Voronoi path

Walking in Delaunay triangulations

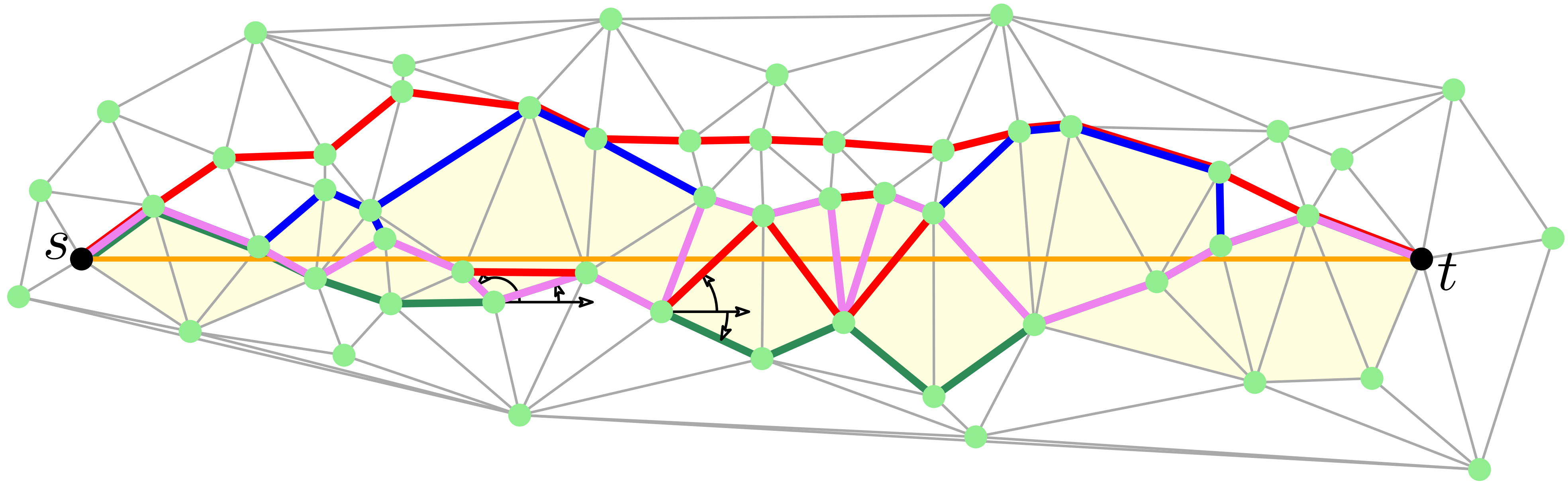
Walk between vertices



Voronoi path with shortcuts

Walking in Delaunay triangulations

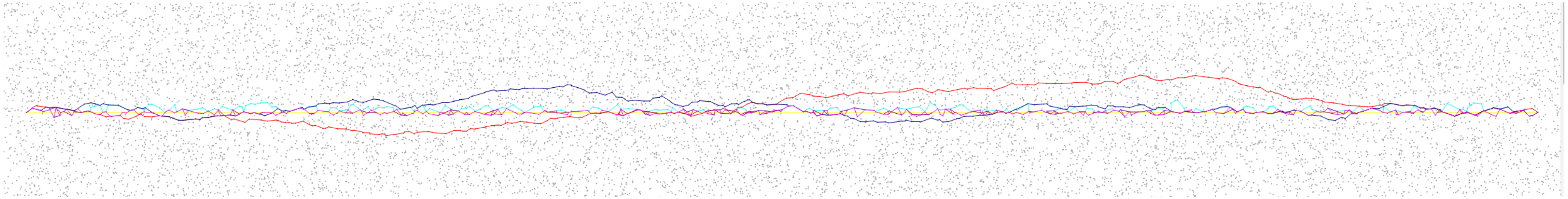
Walk between vertices

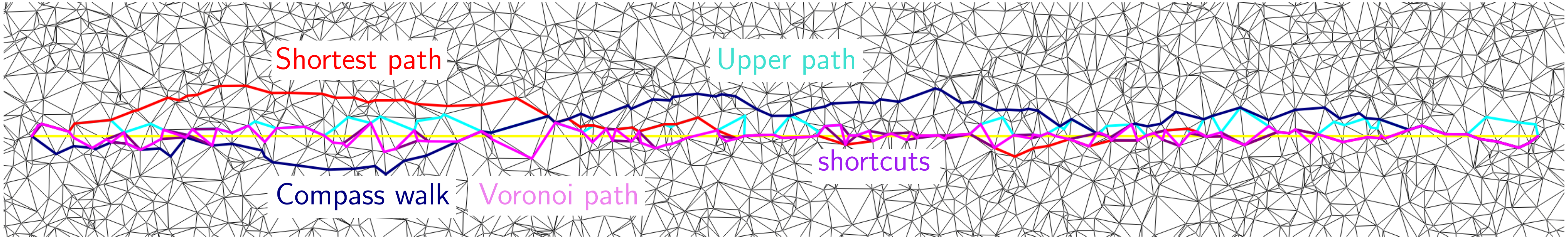


Shortest path
Upper path
Compass walk
Voronoi path with shortcuts

Walking in Delaunay triangulations

Walk between vertices





Walking in Delaunay triangulations

Expected length (experiments)

Euclidean length	1
Shortest path	1.04
Compass walk	1.07
Shortened V. path	1.16
Upper path	1.18
Voronoi path	1.27

Walking in Delaunay triangulations

Expected length (experiments)		theory
Euclidean length	1	
Shortest path	1.04	$\geq 1 + 10^{-11}$
Compass walk	1.07	
Shortened V. path	1.16	1.16 numerical integration
Upper path	1.18	$\frac{35}{3\pi^2} \simeq 1.18$
Voronoi path	1.27	$\frac{4}{\pi} \simeq 1.27$ [Baccelli et al., 2000]

Walking in Delaunay triangulations

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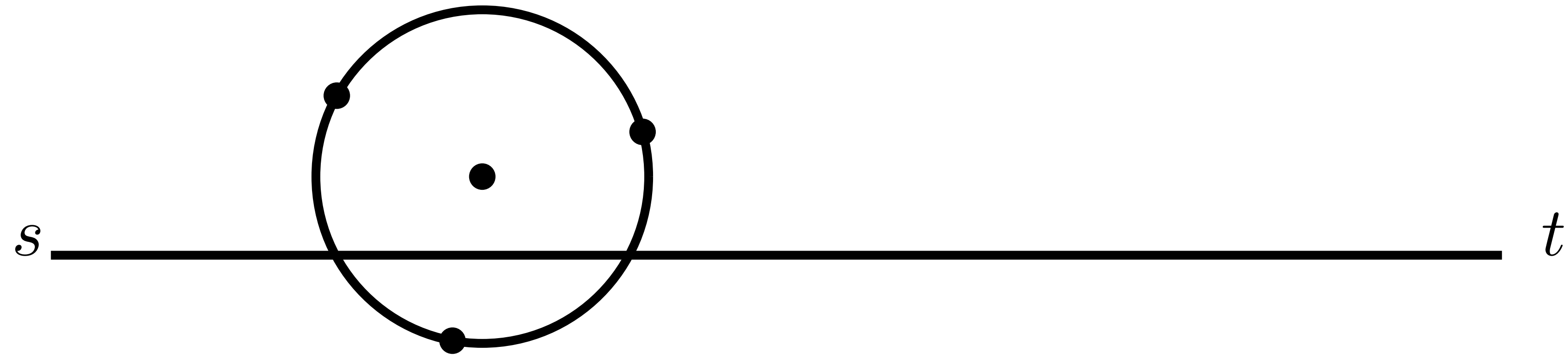
Expected length of upper path

Poisson Delaunay triangulation, rate 1

$$\mathbb{E}\left[length\right] = \mathbb{E}\left[\sum_{triangle \in X_n^3} \mathbb{1}_{[triangle \text{ is Delaunay}]} \mathbb{1}_{[\text{first edge above } st]} \text{length(first edge)}\right]$$



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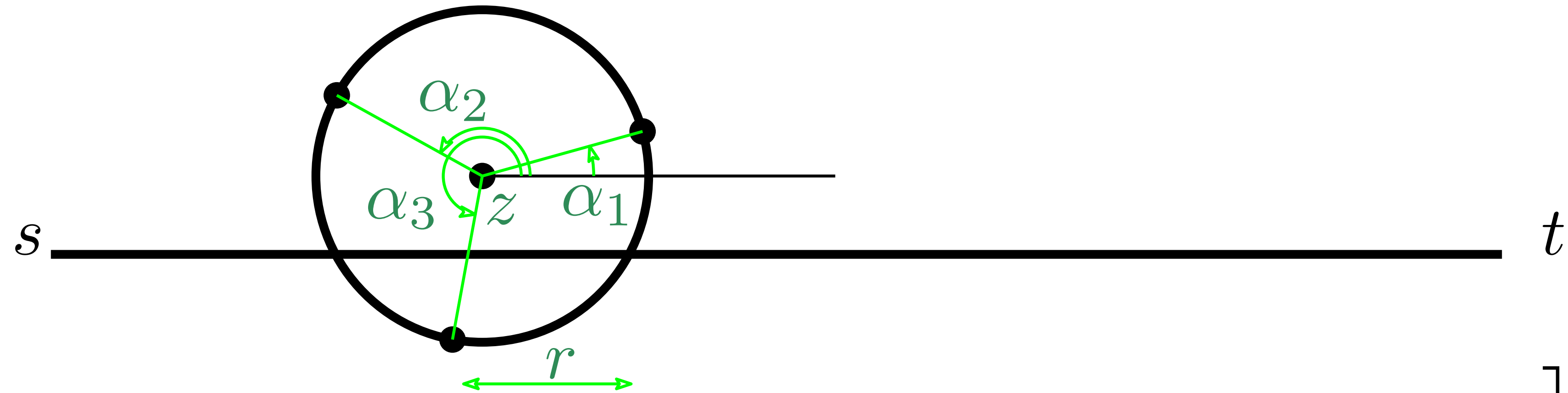
$$= n^3 \int_{(\mathbb{R}^2)^3} \mathbb{P}\left[triangle \text{ is Delaunay}\right] \mathbb{1}_{[\text{first edge above } st]} \text{length(first edge)} \mathrm{d}triangle$$

Slivnyak-Mecke



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$$= n^3 \int_{r=0}^{\infty} \int_{x_z=0}^{1} \int_{y_z=-r}^{r} \int_{[0,2\pi]^3} e^{-n\pi r^2} \mathbb{1}_{[\text{first edge above } st]} 2r \sin\frac{\alpha_1 - \alpha_2}{2} r^3 2\mathcal{A}(triangle) \mathrm{d}\alpha_{1:3} \mathrm{d}y_z \mathrm{d}x_z \mathrm{d}r$$

Blaschke-Petkantschin

Expected length of upper path

Poisson Delaunay triangulation, rate 1

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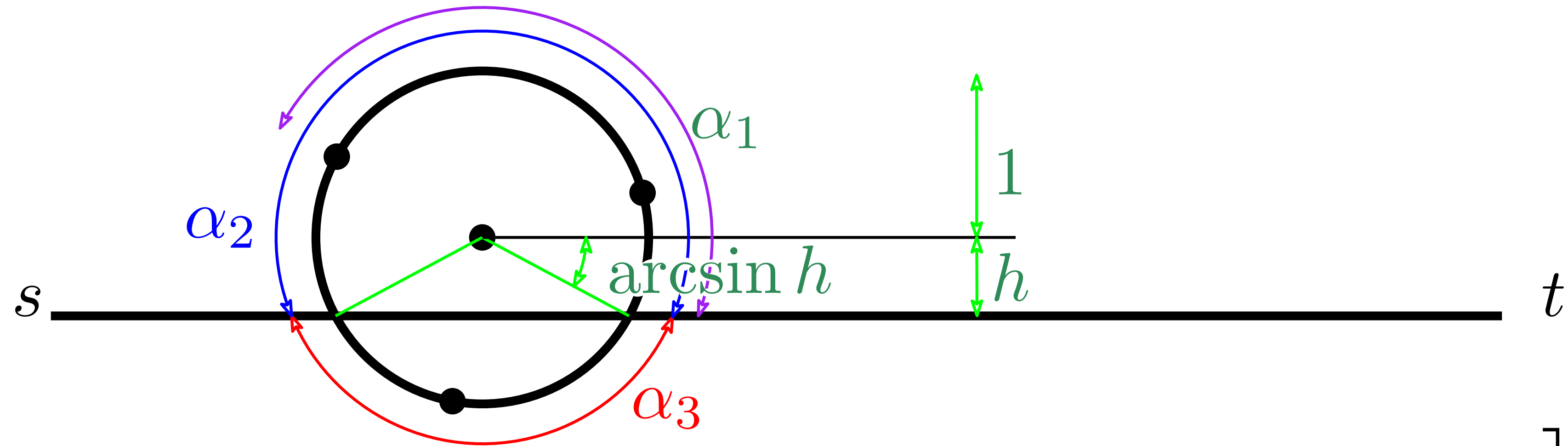
$$= 4n^3 \left(\int_{r=0}^{\infty} e^{-n\pi r^2} r^5 \mathrm{d}r \right) \cdot \left(\int_{h=\frac{y_z}{r}=-1}^{1} \int_{[0,2\pi]^3} \mathbb{1}_{[\text{first edge above } st]} \sin \frac{\alpha_1 - \alpha_2}{2} \mathcal{A}(triangle) \mathrm{d}\alpha_{1:3} \mathrm{d}h \right)$$



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$$= 4n^3\cdot\frac{1}{\pi^3 n^3}\cdot\frac{35\pi}{12} = \frac{35}{3\pi^2}$$

Walking in Delaunay triangulations

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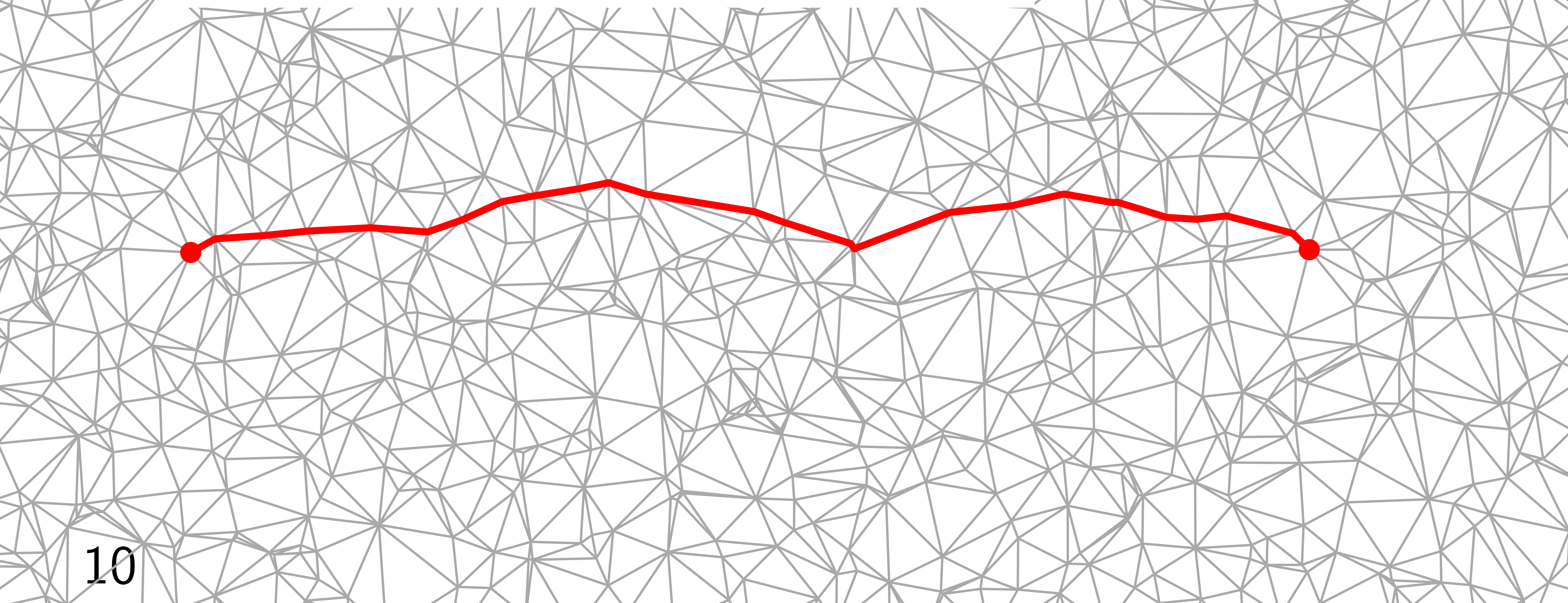
Bad edge = almost horizontal edge

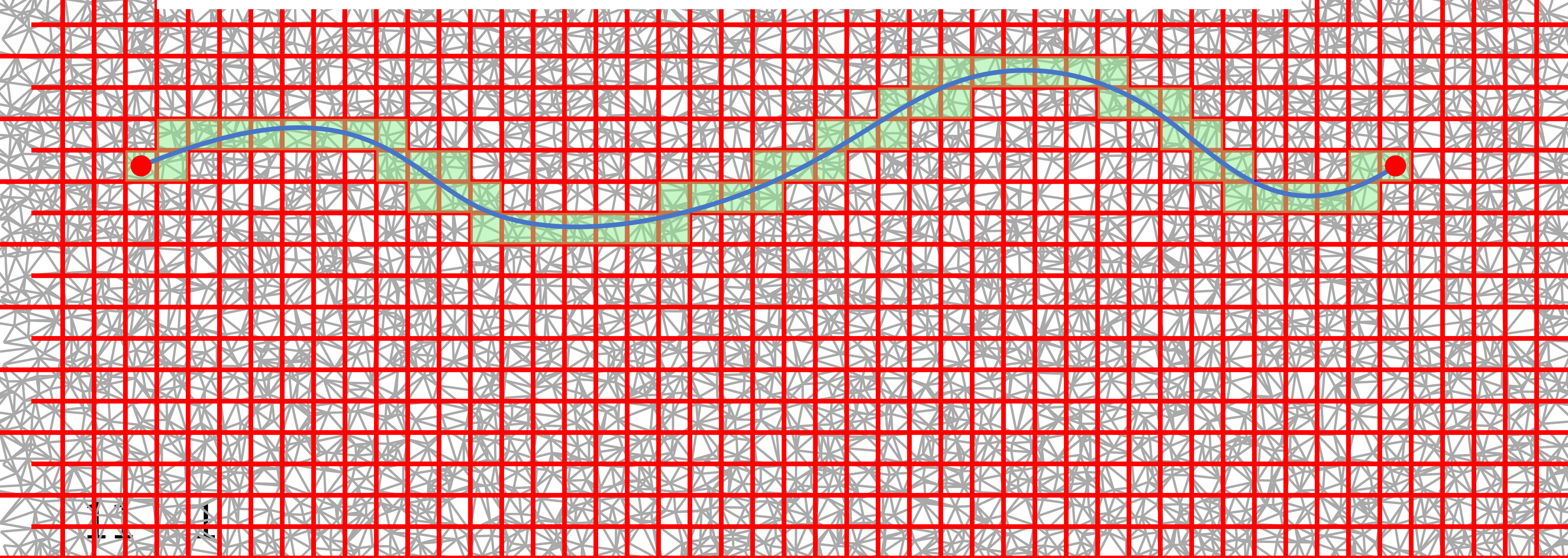
Many bad edges $\Leftarrow$ length close to 1

$\mathbb{P}\left[bad\right]$ = small constant

difficult dependencies to handle



Make a grid



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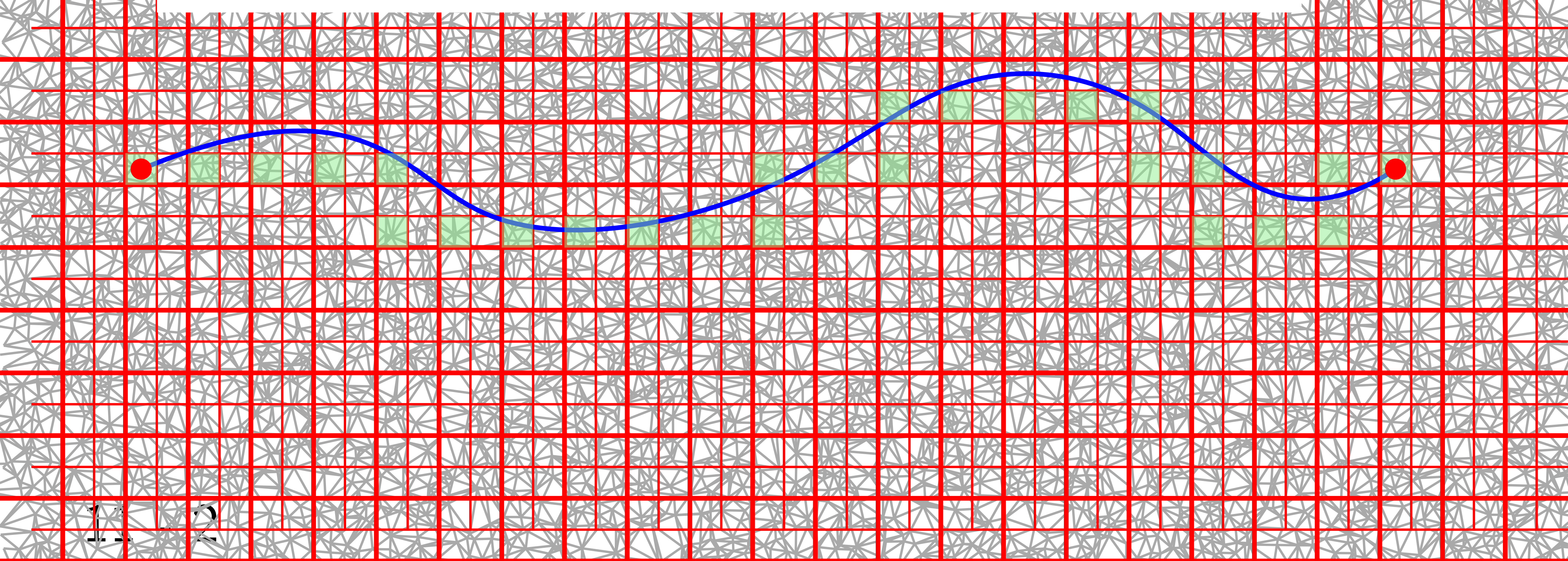
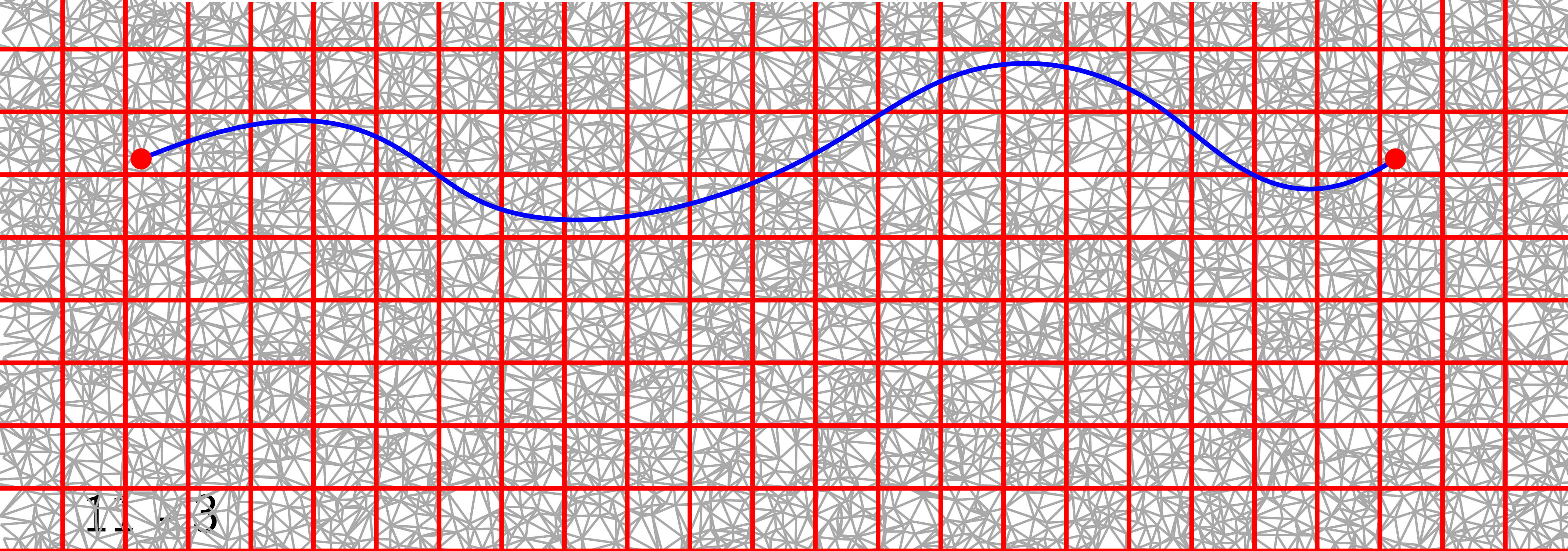
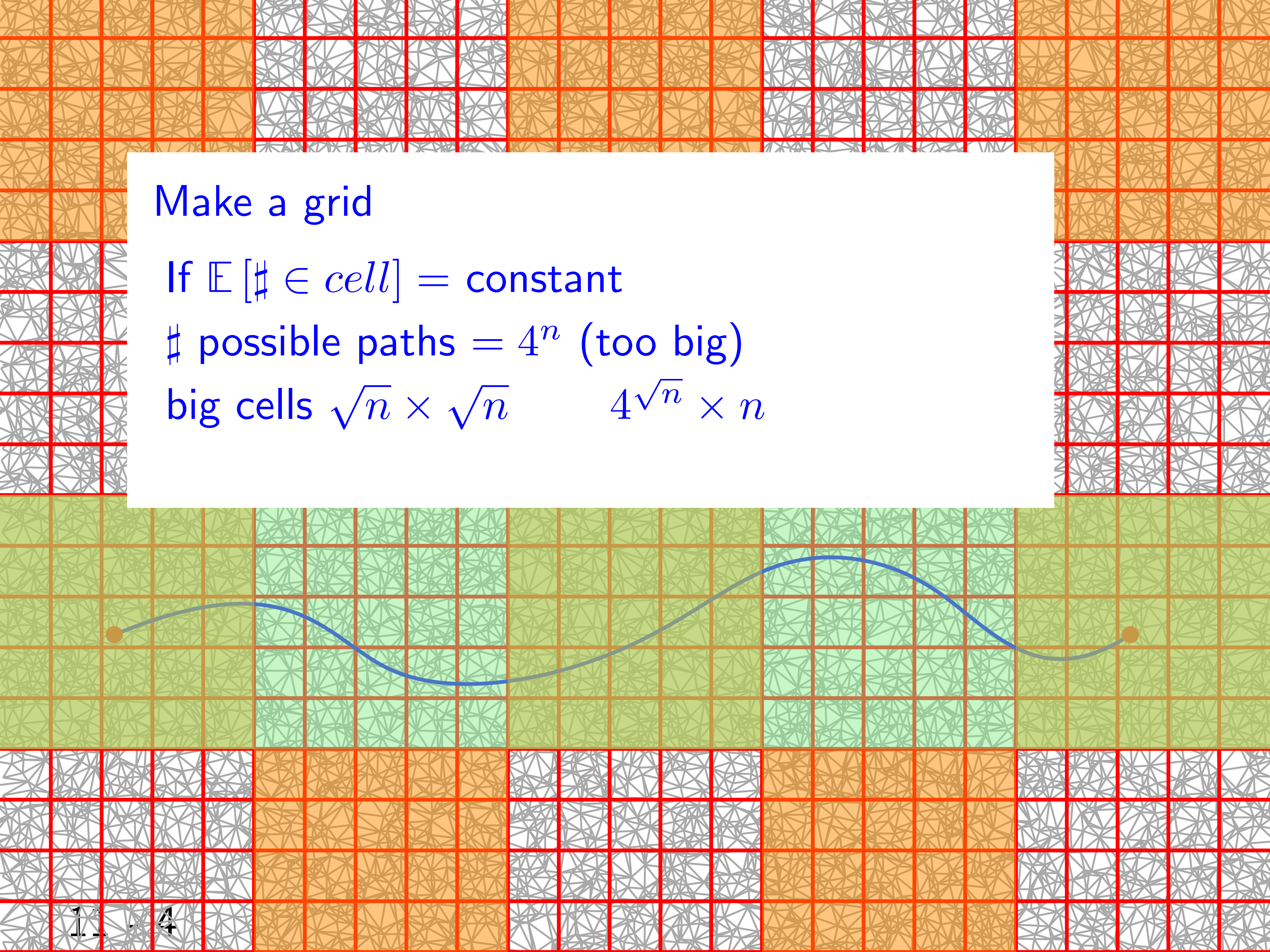


Make a grid

If $\mathbb{E}\left[\sharp \in cell\right] =$ constant

$\sharp$ possible paths $= 4^n$ (too big)





Make a grid

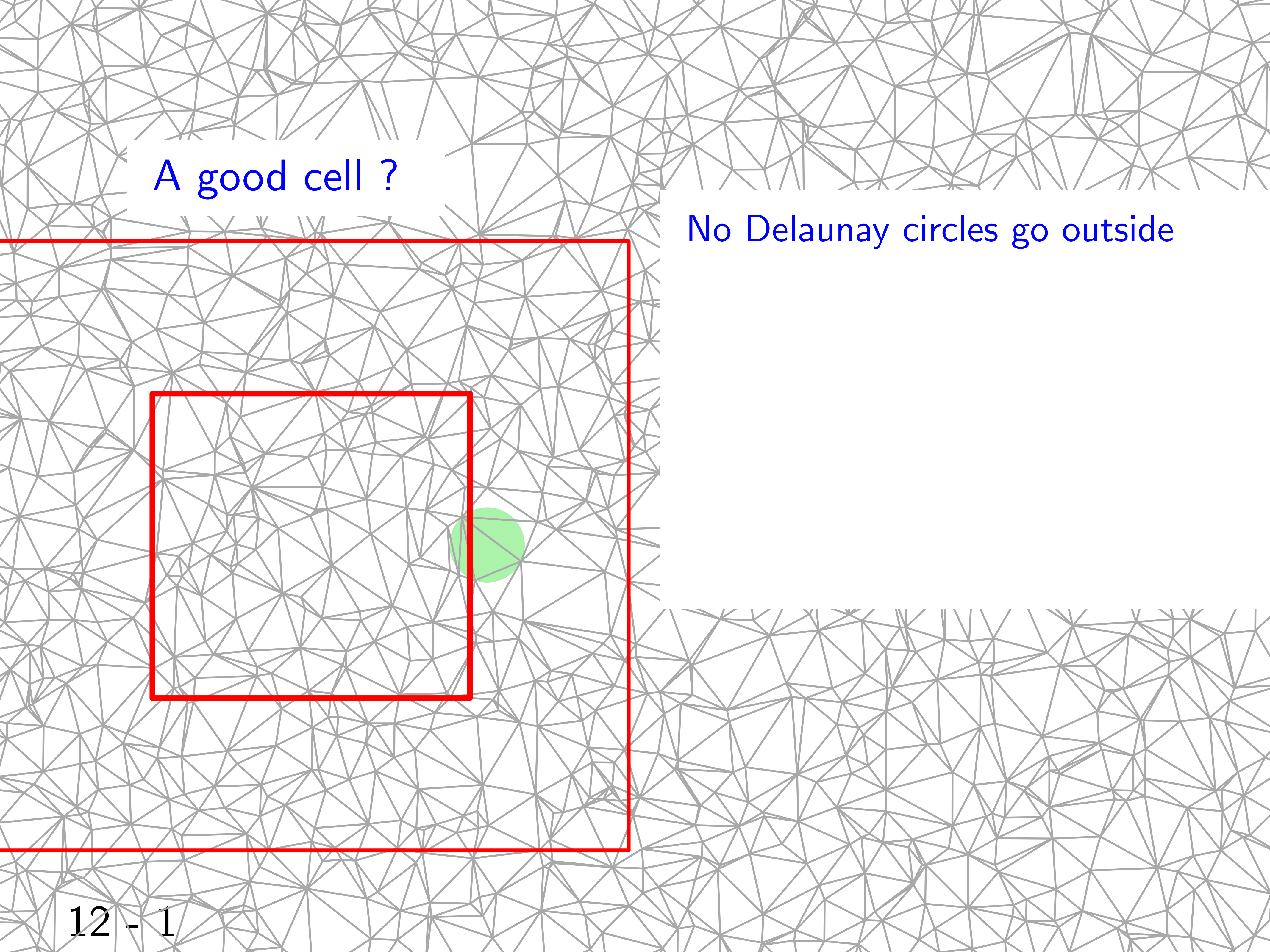
If $\mathbb{E}[\sharp \in cell] =$ constant

$\sharp$ possible paths $= 4^n$ (too big)

big cells $\sqrt{n} \times \sqrt{n}$ $\quad 4^{\sqrt{n}} \times n$

A good cell ?

No Delaunay circles go outside



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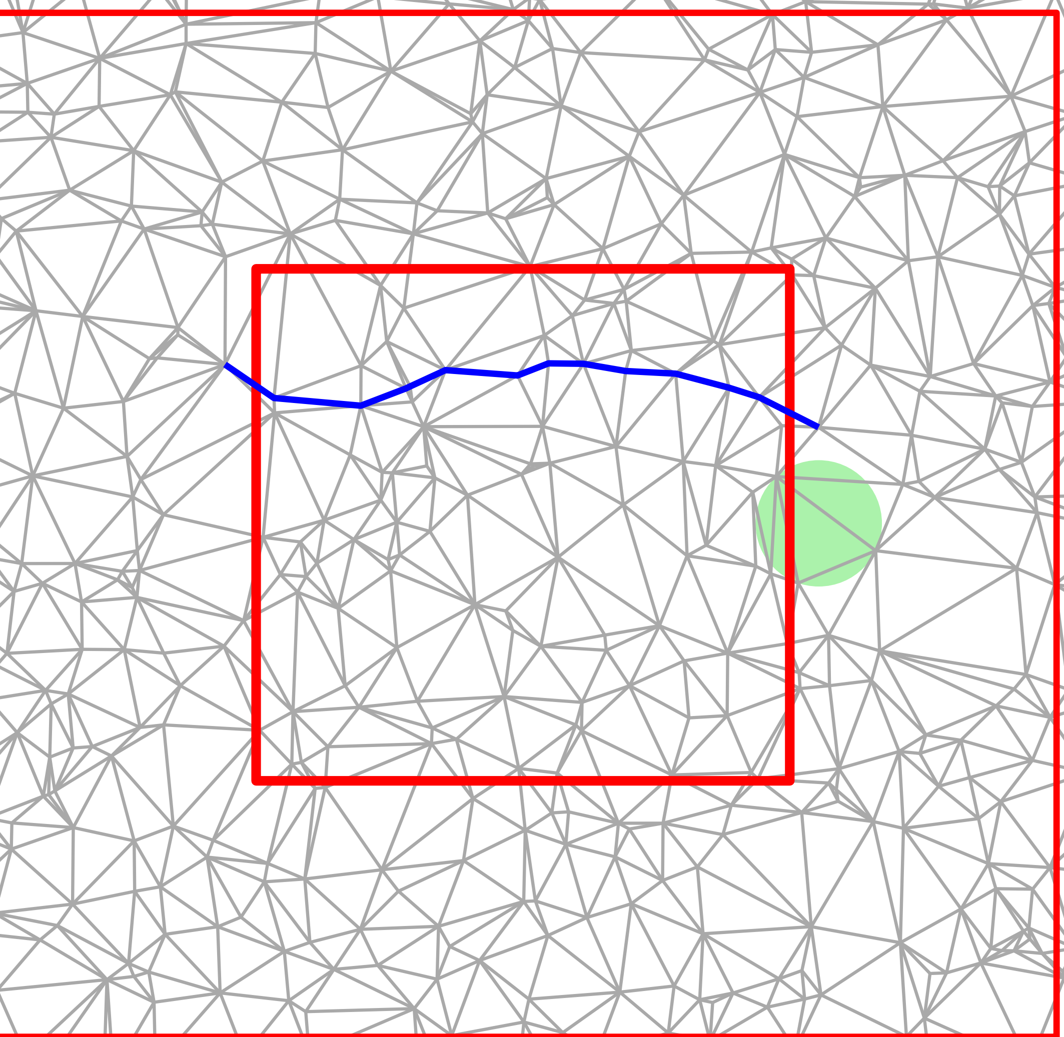
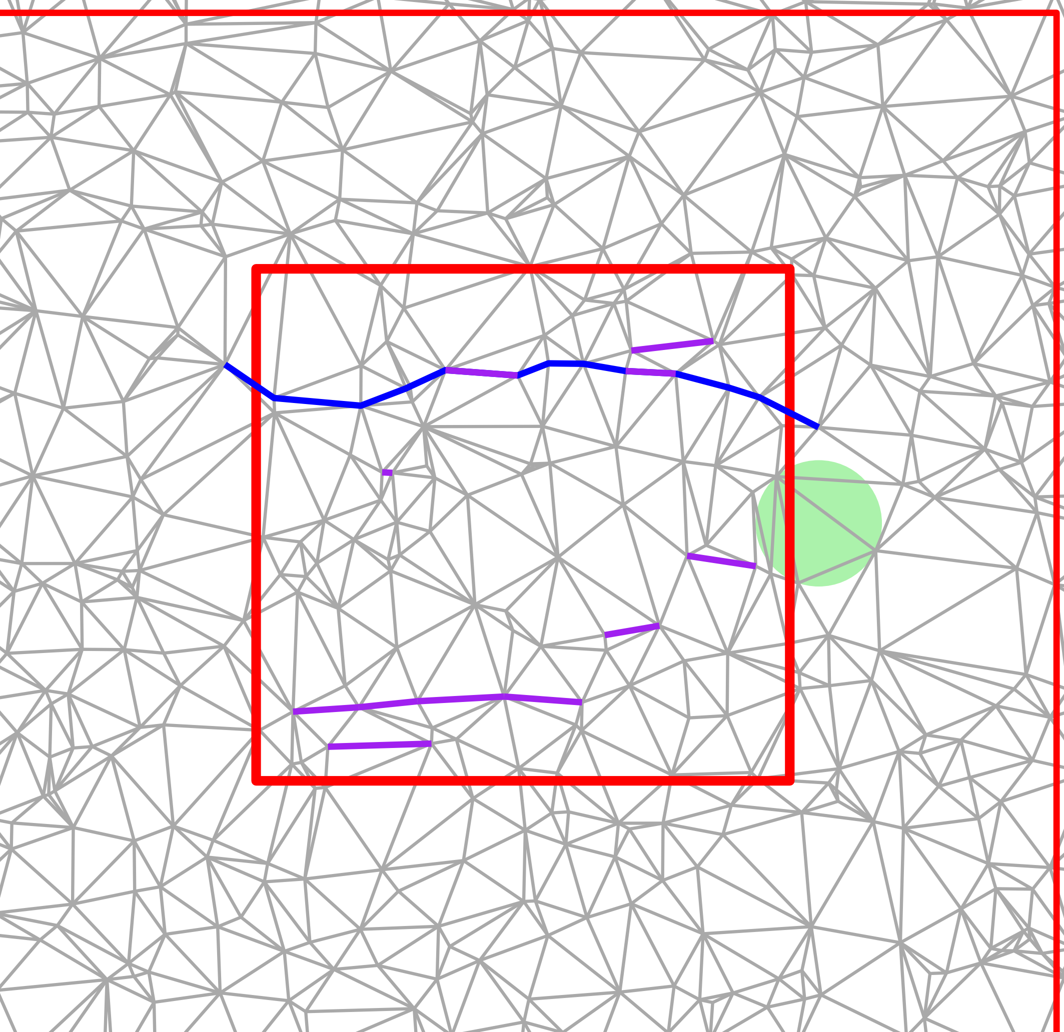


No Delaunay circles go outside

No short path from left to right

A good cell ?

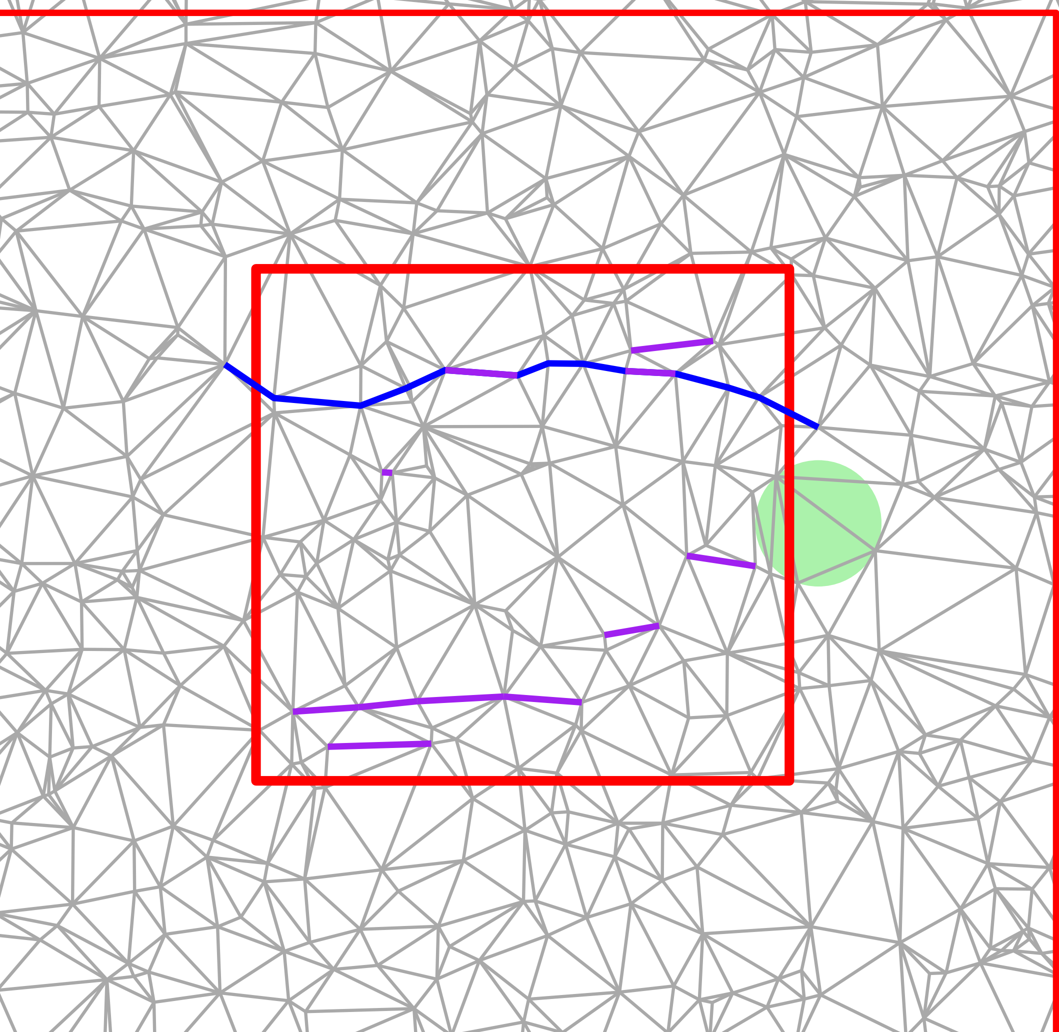


No Delaunay circles go outside

~~No short path from left to right~~

Not enough edges
to make a short path

A good cell ?



No Delaunay circles go outside

~~No short path from left to right~~

Not enough edges
to make a short path

Choose ♯ points in cell, 153

Choose what "short path" means

$1+10^{-10}$

$\mathbb{P}\left[length \geq 1 + 2.5 \times 10^{-11}\right] \leq O\left(\frac{1}{\sqrt{n}}\right)$

Walking in Delaunay triangulations

Voronoi path 1.27 $\frac{4}{\pi} \simeq 1.27$ [Baccelli et al., 2000]

Voronoi path in higher dimension

Walking in Delaunay triangulations

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Voronoi path in higher dimension

$$\mathbb{E}\left[\ell(VP_X)\right] = \frac{1}{2}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{(\mathbb{S}_{d-1})^2} \mathbb{P}\left[B\left((x,0,\ldots,0), r\right) \cap\right.$$

$$\cdot r\|u_1 u_2\|\left|\det(J_\Phi)\right| \mathrm{d}\alpha_{1,1}\ldots\mathrm{d}\alpha_{1,d-1}\mathrm{d}\alpha_{2,1}\ldots$$

Integral form

Walking in Delaunay triangulations

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Voronoi path in higher dimension

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$$\cdot r \|u_1 u_2\| \, |\det(J_\Phi)| \mathrm{d}\alpha_{1,1} \ldots \mathrm{d}\alpha_{1,d-1} \mathrm{d}\alpha_{2,1} \ldots$$

Integral form

Use Taylor expansion to be able to integrate

Walking in Delaunay triangulations

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Voronoi path in higher dimension

$$\frac{\Gamma\left(\frac{d}{2}\right)^4 2^{4d-5} d}{\pi^2 (2d-2)!} \left(1 - \frac{d-1}{4d^2-1}\right)\sqrt{2} \leq \mathbb{E}\left[\ell(VP_X)\right] \leq \frac{\Gamma}{\pi}$$

Walking in Delaunay triangulations

Voronoi path 1.27 $\frac{4}{\pi} \simeq 1.27$ [Baccelli et al., 2000]

Voronoi path in higher dimension

asymptotic behavior between

$$\sqrt{\frac{2d}{\pi}} \quad -\frac{1}{4\sqrt{2d\pi}} + O(d^{\frac{3}{2}})$$

$$+\frac{3}{4\sqrt{2d\pi}} + O(d^{\frac{3}{2}})$$

Walking in Delaunay triangulations

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Voronoi path in higher dimension

asymptotic behavior

$$\sqrt{\frac{2d}{\pi}}$$

Walking in Delaunay triangulations

Voronoi path 1.27 $\frac{4}{\pi} \simeq 1.27$ [Baccelli et al., 2000]

Voronoi path in higher dimension

d	k	lower bound	$\simeq$	exact value	upper bound	$\simeq$
3	41	$\frac{7889842784702576406906971 43}{7450005363375152289126809 60}\sqrt{2}$	1.49770	1.500	$\frac{452337036471251065807696350 9}{426448582869060441377603584 0}\sqrt{2}$	1.50007
4	7	$\frac{102494570}{8729721}\frac{\sqrt{2}}{\pi^2}$	1.6823	1.698	$\frac{121774997}{10270260}\frac{\sqrt{2}}{\pi^2}$	1.6990
5	3	$\frac{135}{104}\sqrt{2}$	1.8357	1.875	$\frac{21305}{16016}\sqrt{2}$	1.8812
6	1	$\frac{3014656}{225225}\frac{\sqrt{2}}{\pi^2}$	1.9179	2.04	$\frac{753664}{51975}\frac{\sqrt{2}}{\pi^2}$	2.0778
7	1	$\frac{210}{143}\sqrt{2}$	2.0768	2.2	$\frac{225}{143}\sqrt{2}$	2.2252
8	1	$\frac{2080374784}{134008875}\frac{\sqrt{2}}{\pi^2}$	2.2244	2.3	$\frac{130023424}{7882875}\frac{\sqrt{2}}{\pi^2}$	2.3635

numerical integration

Thank you