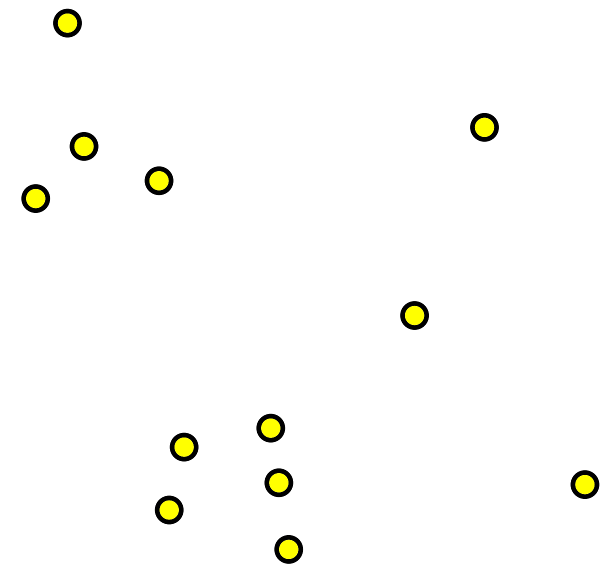
Shatter functions of (geometric) hypergraphs

(c) Ori Gersht

Boris Bukh & Xavier Goaoc

arXiv:1701.06632

Vapnik-Chervonenkis dimension via an example.



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$\mathcal{H}(P) =$ all possible intersections
with a (closed) half-plane

$\mathcal{H}(P) \subseteq 2^P$, the set of all subsets of P (including $\emptyset$).
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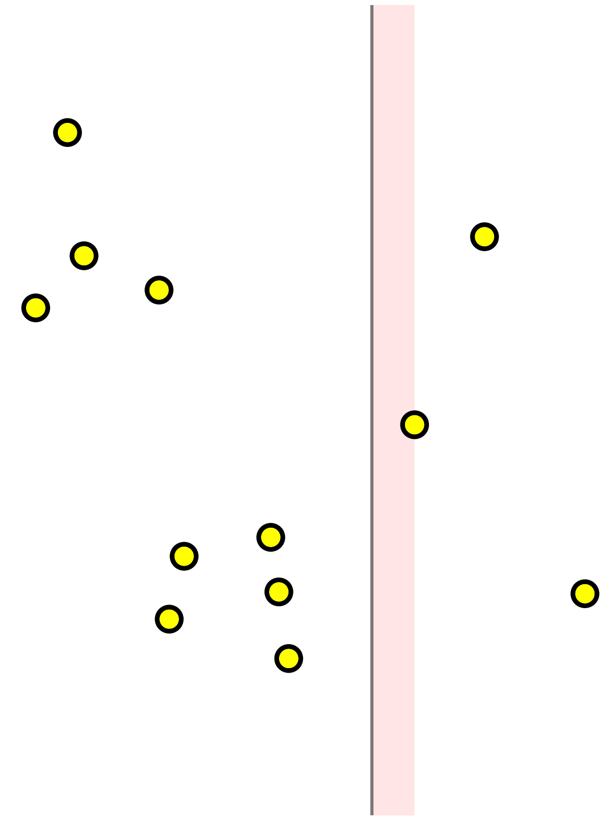

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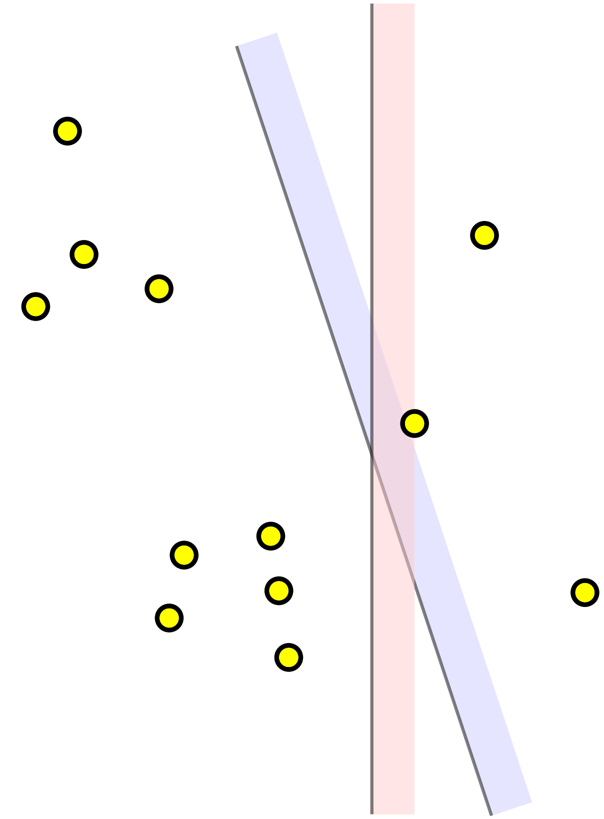

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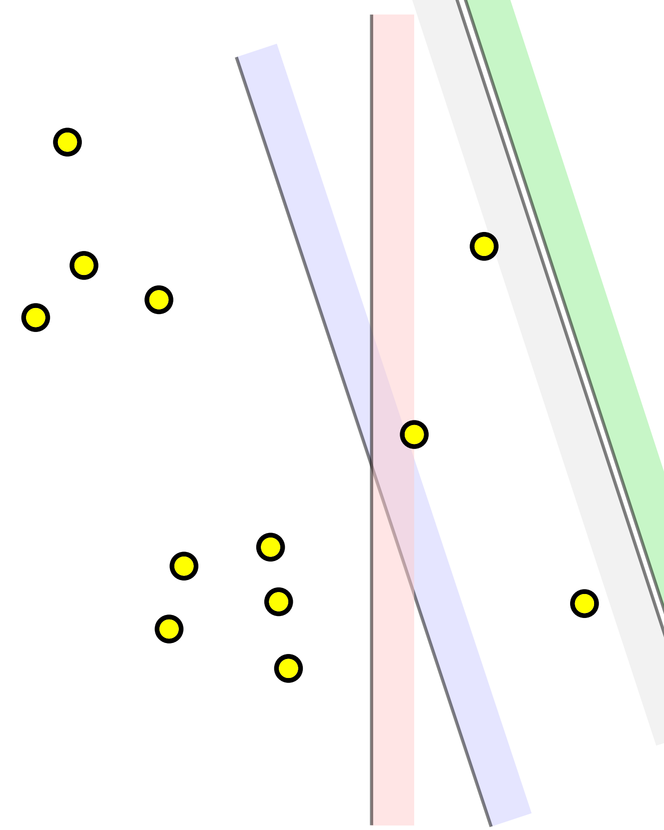

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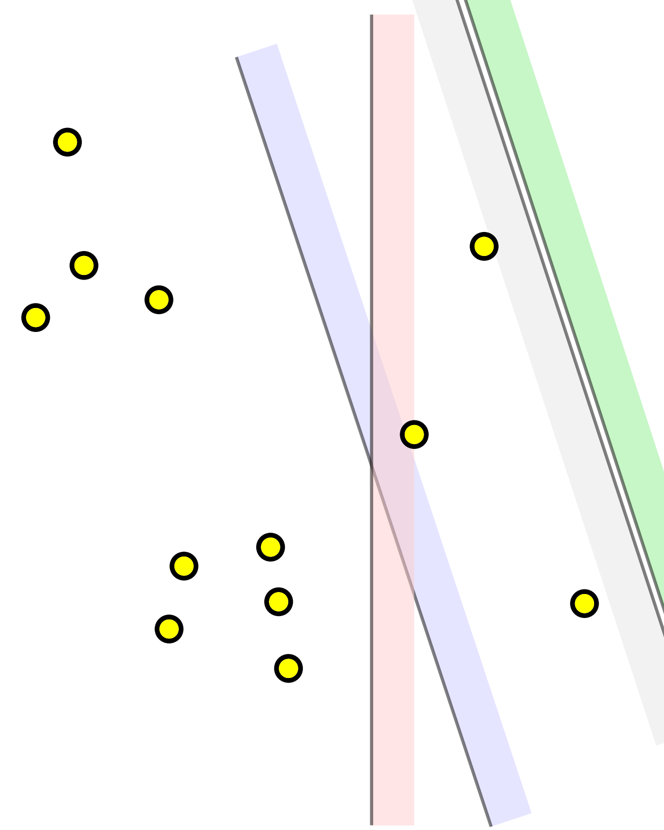

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How large is $\max_{|P|=n} |\mathcal{H}(P)|$?

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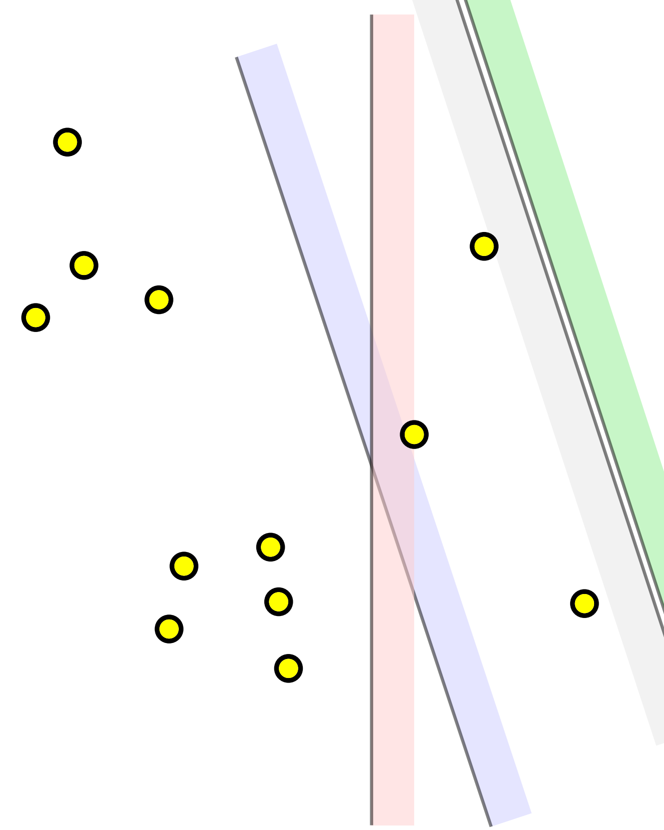

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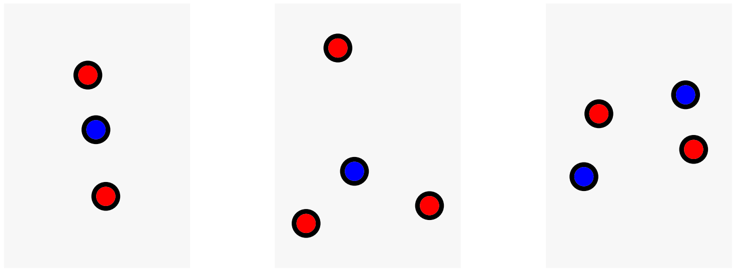

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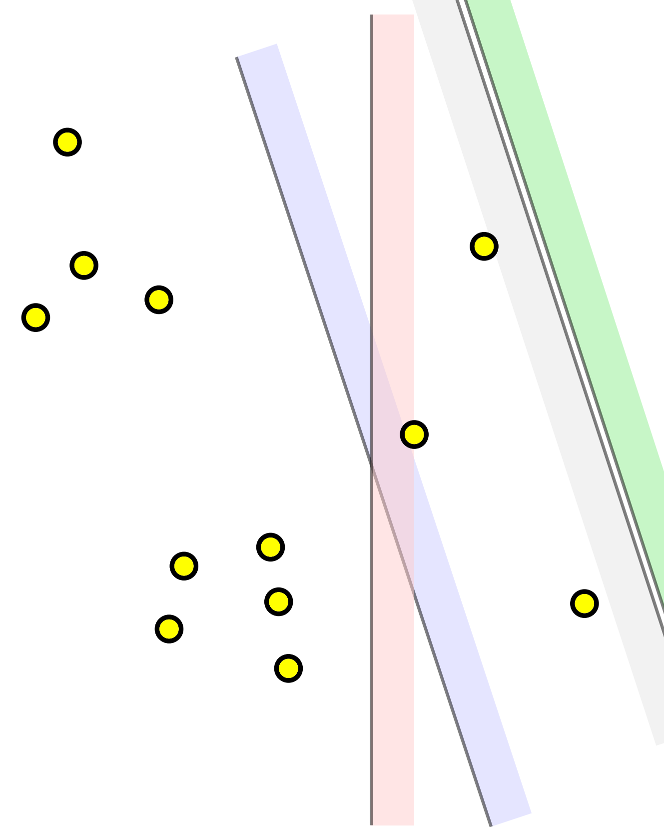

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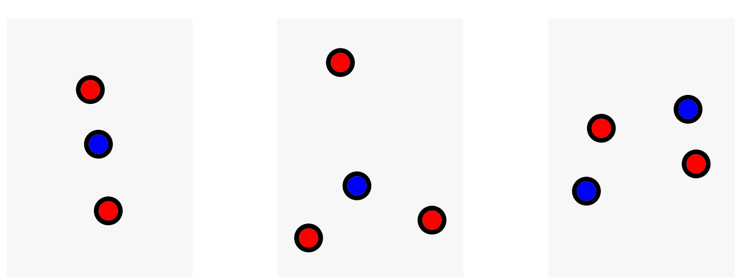

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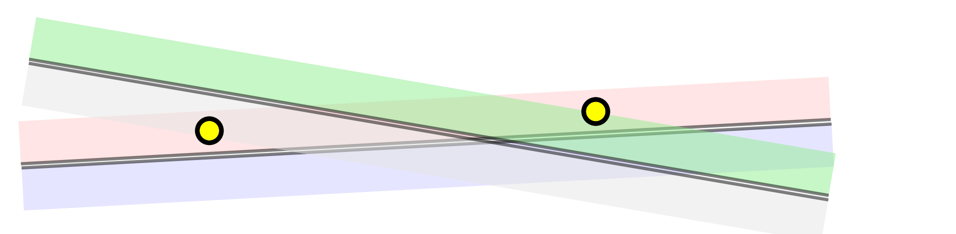

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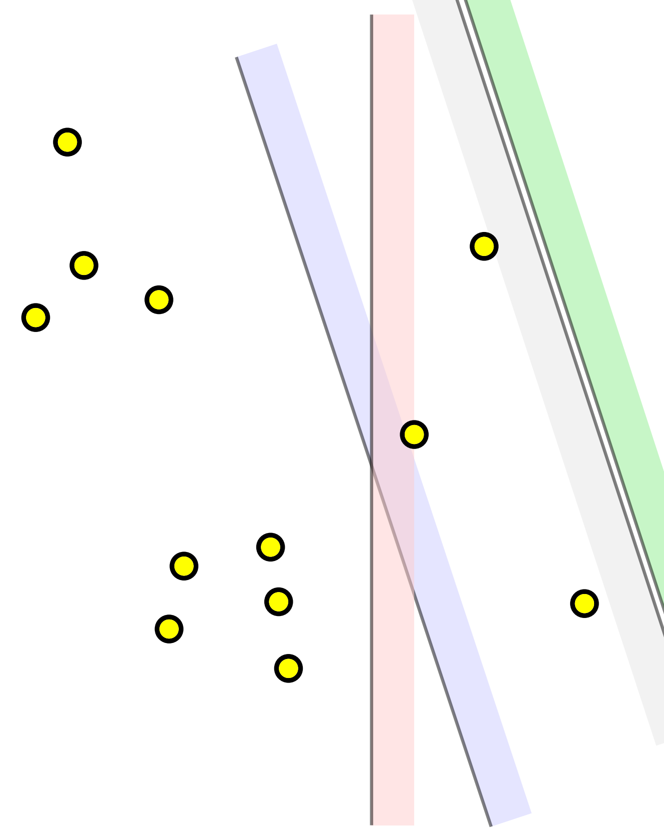

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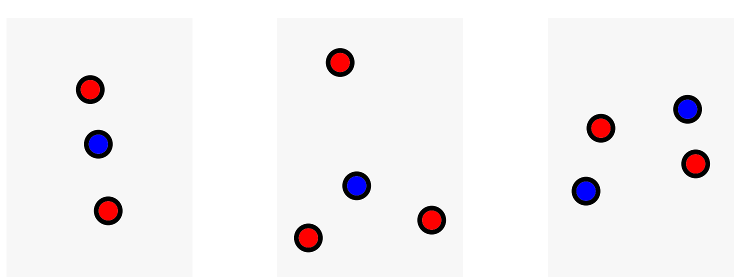

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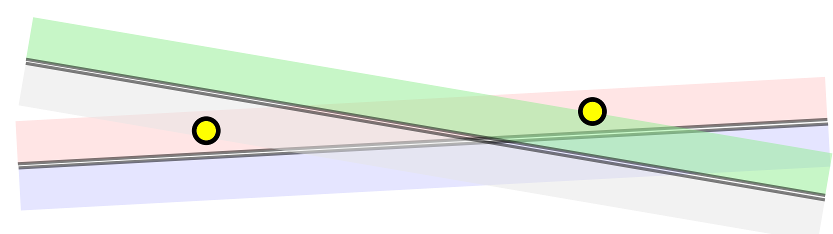

The two questions are related at a combinatorial level.

Vapnik-Chervonenkis dimension, formally.

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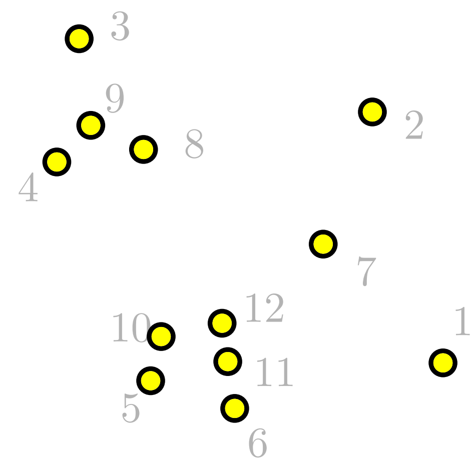

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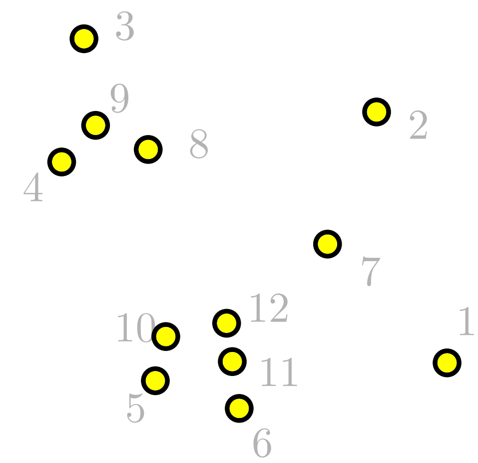

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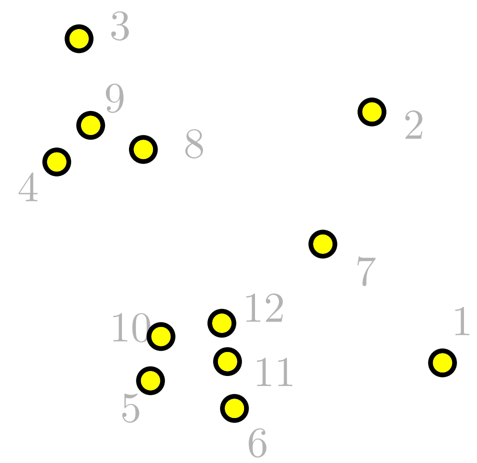

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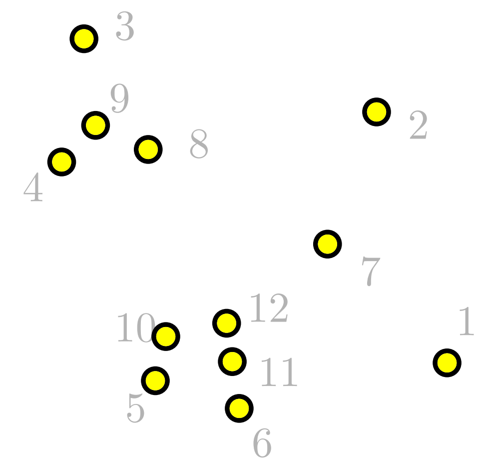

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Sauer's lemma. $f_{\mathcal{H}}(k+1) < 2^{k+1} \quad \Rightarrow \quad f_{\mathcal{H}}(n) \le \sum_{i=0}^{k} \binom{n}{i} = O(n^k)$.

[Vapnik-Chervonenkis'71][Sauer'72][Shelah'72].

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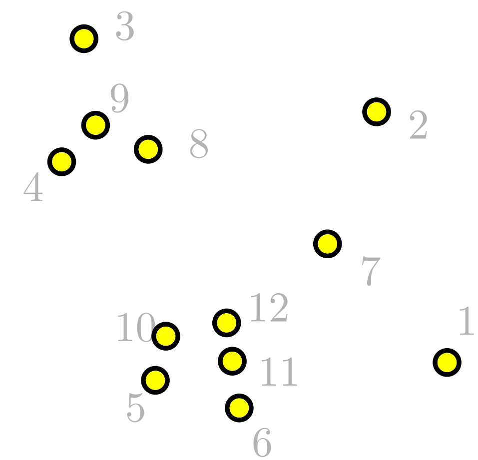

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[Vapnik-Chervonenkis'71][Sauer'72][Shelah'72].

Exponential/polynomial dichotomy.

The VC-dimension of $\mathcal{H}$ is $\max\{k : f_{\mathcal{H}}(k) = 2^k\}$.

Examples of applications to HITTING-SET problem:

Given: sets $A_1, A_2, \dots, A_n \subset X$ ($\simeq$ a hypergraph)

Find: a smallest $Y \subseteq X$ s.t. $A_i \cap Y \neq \emptyset$ for $i = 1, 2, \dots, n$

Better bound on the approximation ratio of the greedy algorithm.

$O(\log |opt|)$ in place of $O(\log n)$ where $n =$ number of sets.

ϵ-net theorem.

$O(\frac{d}{\epsilon} \log \frac{1}{\epsilon})$ points suffice to hit all sets of size $\geq \epsilon n$.

ϵ-approximation theorem.

$O(\frac{d}{\epsilon^2} \log \frac{1}{\epsilon})$ points suffice to approximate all sets of size $\geq \epsilon n$.

fractional Helly theorem

If $\geq \alpha \binom{n}{d+1}$ of the $(d+1)$-element subsets intersect,

then a proportion $\geq \beta(\alpha, d)$ of the sets intersect.

(p, q)-theorem

If among any p sets some q intersect,

then some $c(p, q, d)$ points suffice to hit all sets.

Usually stated with $d \leq$ (dual) VC-dimension but really uses $f_{\mathcal{H}}(n) = O(n^d)$.

Bounding **one** value of $f_{\mathcal{H}}$ restricts its **asymptotic growth**.

$$f_{\mathcal{H}}(k) < 2^k \quad \Rightarrow \quad f_{\mathcal{H}}(n) = O(n^{k-1})$$
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Define $t_k(m)$ as the largest integer such that for any hypergraph $\mathcal{H}$,

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$t_k(m) = \Omega(m^k)$ as $m \to \infty$ conjectured:

Bondy-Hajnal conjecture. For any m, k $\exists n_0(m, k)$ such that

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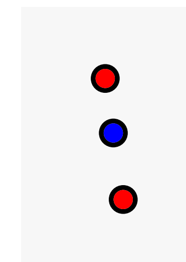

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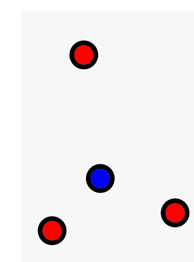
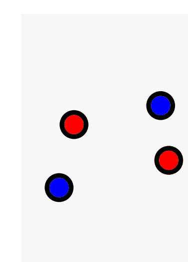


For halfplanes in $\mathbb{R}^2$, $f_\mathcal{H}(3) = 8$ and $f_\mathcal{H}(4) \le 14$.
$\Rightarrow$ Sauer's lemma gives only $O(n^3)$



Does any shatter condition give $O(n^2)$ for points and halfplanes in $\mathbb{R}^2$?

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Known facts...

Compression lemma. For every hypergraph $\mathcal{H}$ there exists a simplicial complex $\mathcal{K}$ such that $|\mathcal{K}| = |\mathcal{H}|$ and $f_{\mathcal{K}} \leq f_{\mathcal{H}}$.

[Alon'83][Frankl'83]

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$$f_{\mathcal{H}}(4) \le 8 \quad \Rightarrow \quad f_{\mathcal{H}}(n) = O(n\sqrt{n})$$
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$t_k(m) \geq 2^k m - (k-1)2^k - 1$ [Cheong-Goaoc-Nicaud'13]

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New results:

Improved shatter condition: $t_k(m) > (2^{k+1} - k - 1)m - 2^{4k}$

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Does any shatter condition give $O(n^2)$ for points and halfplanes in $\mathbb{R}^2$?

No

Some proofs...

Let's prove that $f_{\mathcal{H}}(4) \leq 1 + 4 + \binom{4}{2} = 11 \quad \Rightarrow \quad |\mathcal{H}| \leq 1 + n + \binom{n}{2}$.

B-H for $(k, m) = (2, 4)$

$\mathcal{H}$ consists of $\emptyset$, n vertices, some edges, some triangles.

Compression lemma allows to consider $\mathcal{H}$ as a simplicial complex.

If $Q \in \mathcal{H}$ with $|Q| = 4$ then $f_{\mathcal{H}}(4) \geq |\mathcal{H}_{|Q}| = 16$.

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Add triangles and delete edges so that: $\begin{cases} \text{for every triangle, the } 3 \text{ edges remain,} \\ \text{on any } 4 \text{ vertices, } \# \text{ added triangles} \leq \# \text{ deleted edges.} \end{cases}$

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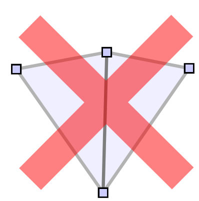

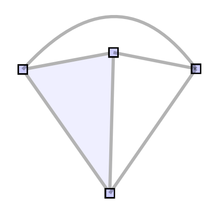

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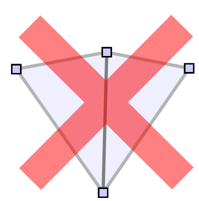

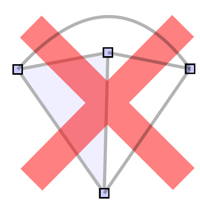

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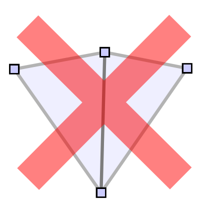

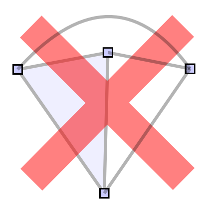

Contradictory incentives:

more triangles $\Rightarrow$ fewer edges,

in each triangle , the degree sum to at most $2 + 2(n-3)$

fewer edges $\Rightarrow$ fewer triangles

at each vertex, # triangles $\leq \frac{\text{\# edges}}{2}$

Proof of the upper bound...

$t_k(m) \leq (2^{k+1} - k - 1)m + 1$

$\Leftrightarrow \exists$ hypergraphs on n vertices with $f_{\mathcal{H}}(m) \leq (2^{k+1} - k - 1)m + 1$ and size $\omega(n^k)$

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Ex.: The Ramsey number $R(k, k)$ is at least $2^{k/2}$.

Uniform random 2-coloration of the edges of K_n.
Probability that k vertices span a monochromatic subgraph $\leq \frac{2}{2^{\binom{k}{2}}}$.
Probability that coloring has a monochromatic k-set $\leq \binom{n}{k} 2^{-\binom{k}{2}+1}$
Look for the largest k such that $\binom{n}{k} 2^{-\binom{k}{2}+1} < 1$

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Tune α so that $\mathcal{C}$ is large **and** $f_{\mathcal{C}}(m)$ is small.

Incompatible conditions...
... but when $\mathcal{C}$ starts to be large, **few** m-tuples have large trace.

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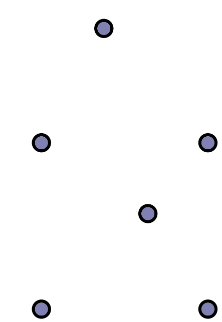

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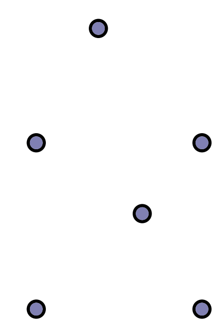

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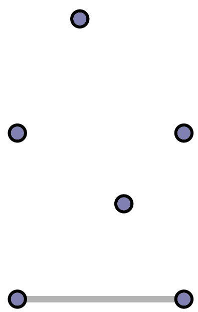

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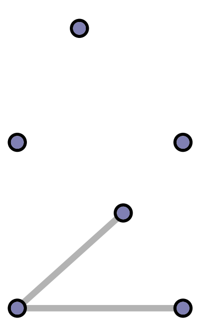

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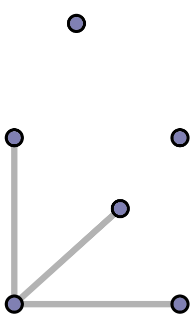

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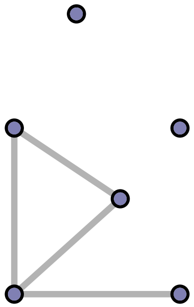

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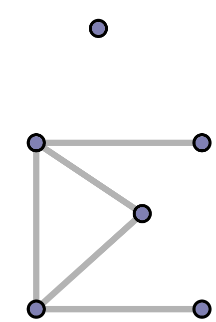

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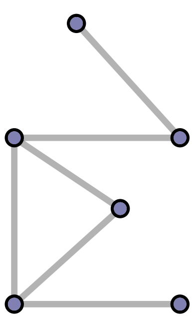

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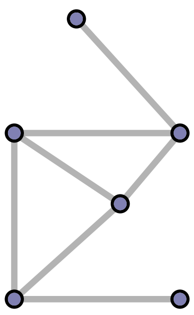

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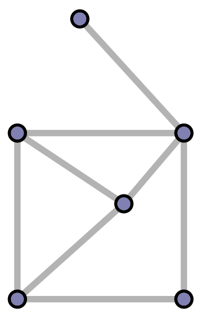

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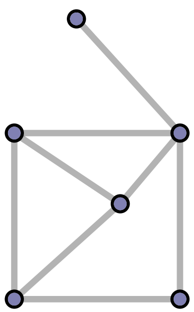

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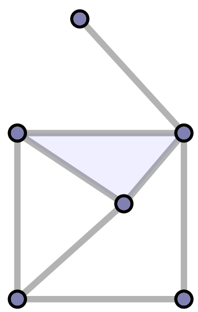

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Start with the empty set and every vertex.

Consider every edge in turn, and add it with probability p. Events are **independant**.

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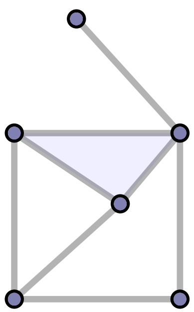

Proof of the upper bound.

$t_k(m) \leq (2^{k+1} - k - 1)m + 1$

$\Leftrightarrow \exists$ hypergraphs on n vertices with $f_{\mathcal{H}}(m) \leq (2^{k+1} - k - 1)m + 1$ and size $\omega(n^k)$

Two-stage probabilistic construction.

Random simplicial complex $\mathcal{C}$ governed by a parameter $p = n^{-\alpha}$.

Tune α so that $\mathcal{C}$ is large **and** $f_{\mathcal{C}}(m)$ is small.

Incompatible conditions...
... but when $\mathcal{C}$ starts to be large, **few** m-tuples have large trace.

Deleting these m-tuples ensure $f_{\mathcal{C}}(m)$ small and does not affect the size much.

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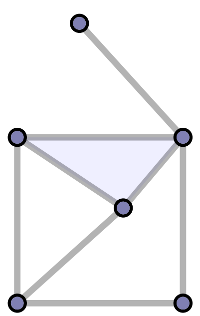
Delete every m-tuple of vertices supporting too many simplices.



Model

n vertices, parameter $p = n^{-\alpha}$

build $\mathcal{C}$ by:

declaring $\emptyset$ and all vertices in $\mathcal{C}$,

examining subsets of size up to $k+1$
in order increasing w.r.t. inclusion,

when examining U, if all proper subsets
are in $\mathcal{C}$, add U to $\mathcal{C}$ with probability p.

parameter z

build $\mathcal{C}'$ by:

a m-tuple V is **bad** if $|\mathcal{C}_{|V}| > 1 + m + z$

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$f_{\mathcal{C}'}(m) \leq 1 + m + z$

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Works for $z = (2^{k+1} - k - 2)m$

Not just hypergraphs...

Condorcet's paradox in voting systems.

Ask each voter to rank (=order) the candidates.

The majority rule may not combine into an order.

$123 : 1/3,\ 231 : 1/3,\ 312 : 1/3$

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How large can a consistent set of permutations on $[n]$ be?

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Shatter function of a family of orders on $[n]$.

Restriction of an order: induced order.

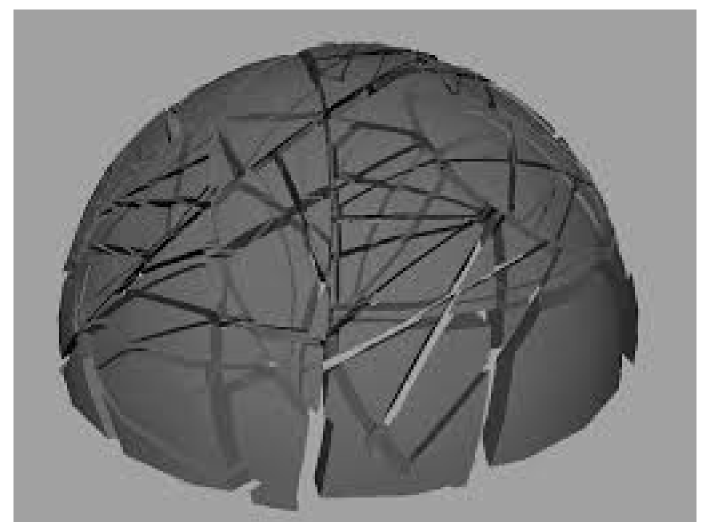
$\mathcal{F}$ a family of orders.

$f_{\mathcal{F}}(k) =$ the maximum number of restrictions on a k-element subset.

Consistent $\Rightarrow f_{\mathcal{F}}(3) < 6 \Rightarrow |\mathcal{F}|$ at most exponential in n.

[Raz'00]

Thank you for your attention!



A few words on the lower bound...

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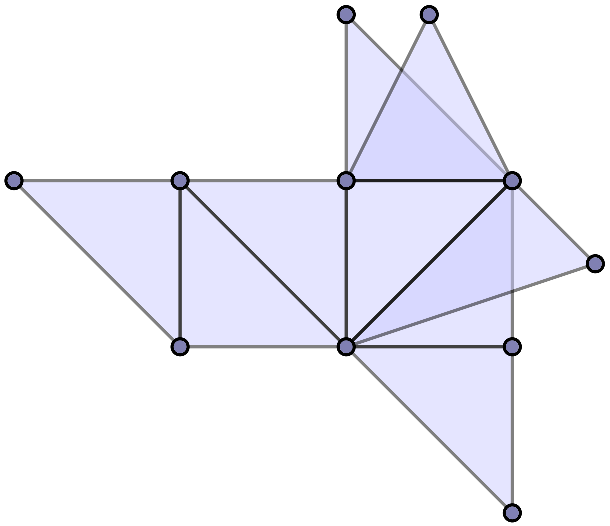
Ingredients:

d-dimensional trees, degree of a $(d-1)$-dimensional simplex



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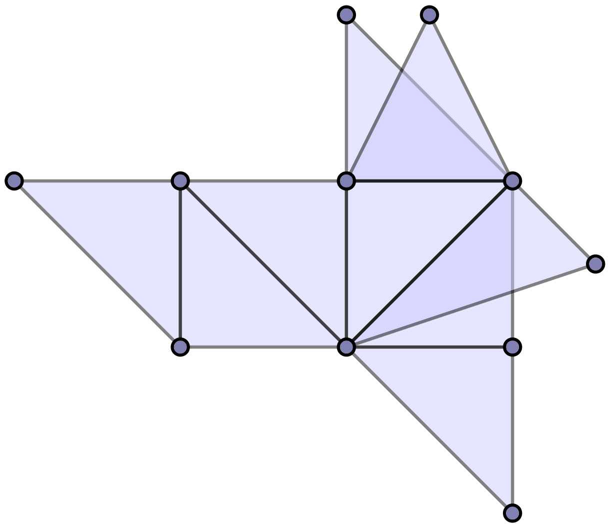

d-dimensional trees, degree of a $(d-1)$-dimensional simplex

Core of the argument:

Analysis proceed by increasing dimension.

Many d-dimensionals simplices $\Rightarrow$ subcomplex with all $(d-1)$-simplices of high degree

$\Rightarrow$ many d-trees that can be combined to find a large trace

Trees have a prescribed density (#simplices / #vertices) and allow combination ("balanced").

Adaptation of a technique of Bukh-Conlon (edge density in graphs with forbidden patterns).