The Computational Geometry of Congruence Testing

Part I. Testing two geometric objects for congruence, i.e., whether they are the same up to translations and rotations (and possibly reflections) is a fundamental question of geometry.

In the first part, I will survey the various algorithmic techniques that have been used since the 1970s to solve the problem in two and three dimensions in $O(n \log n)$ time for two n-point sets, such as string matching, planar graph isomorphism (Sugihara [6]), and the reduction technique of Atkinson [3].

In d-dimensions, for small constant d, the best previous algorithm takes $O(n^{\lceil d/3 \rceil} \log n)$ time (Brass and Knauer [4]). There is also a randomized Monte Carlo algorithm of Akutsu [1] and Matoušek, which takes $O(n^{\lfloor d/2 \rfloor /2} \log n)$ time but which may miss to find a congruence, with small probability. I will review the involved techniques: the basic dimension reduction technique of Alt, Mehlhorn, Wagener, and Welzl [2], the canonical forms of Akutsu [1], the closest-pair graph of Matoušek.

Part II. In the second part, I will introduce our recent algorithm for solving the 4-dimensional problem in $O(n \log n)$ time (joint work with Heuna Kim [5]). This algorithm will require the study of four-dimensional geometry, in particular the structure of four-dimensional rotations, Hopf fibrations, and the regular polytopes.

References

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