# Adjoint Map Representation for Shape Analysis and Matching

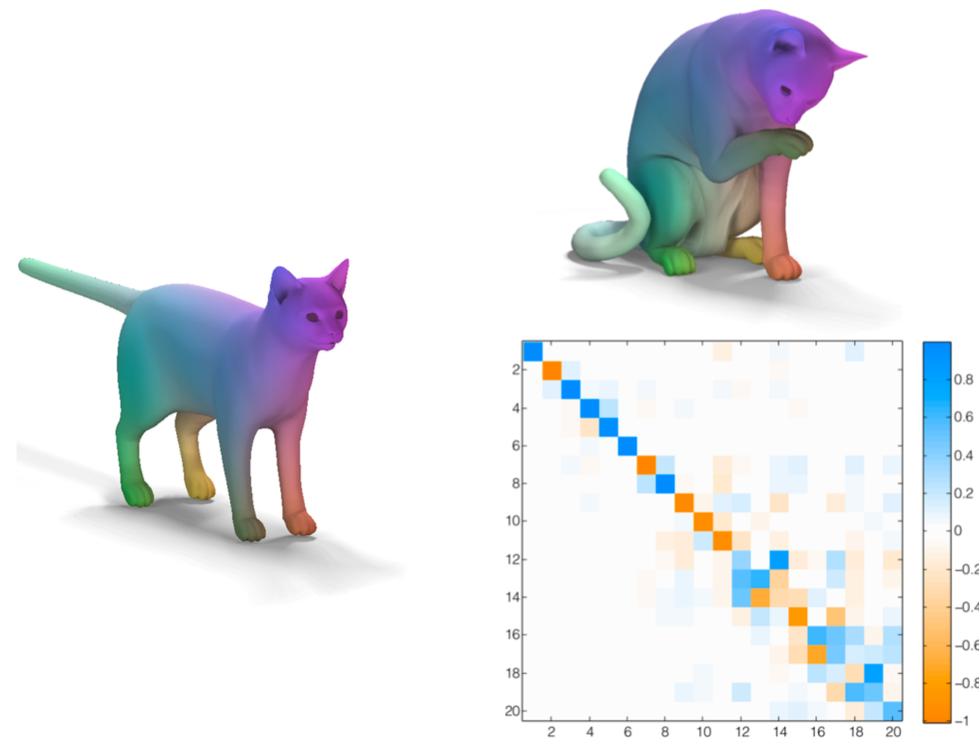
**Ruqi Huang**, LIX

Joint work with *Maks Ovsjanikov*.

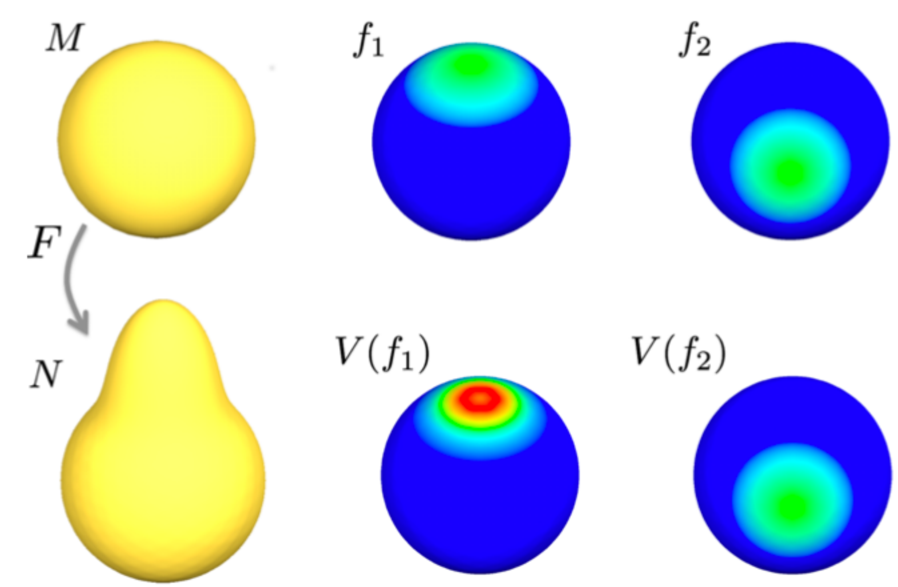
14 Decmember @ JGA, 2017

# Motivation and Target

- Operator-based representations.



Functional maps[1]




Shape difference[2]

1 *Functional Maps: a Flexible Representation of Maps between Shapes*, M. Ovsjanikov et al., SIGGRAPH, 2012;
2 *Map-based Exploration of Intrinsic Shape Differences and Variability*, R. Rustamov et al., SIGGRAPH 2013.

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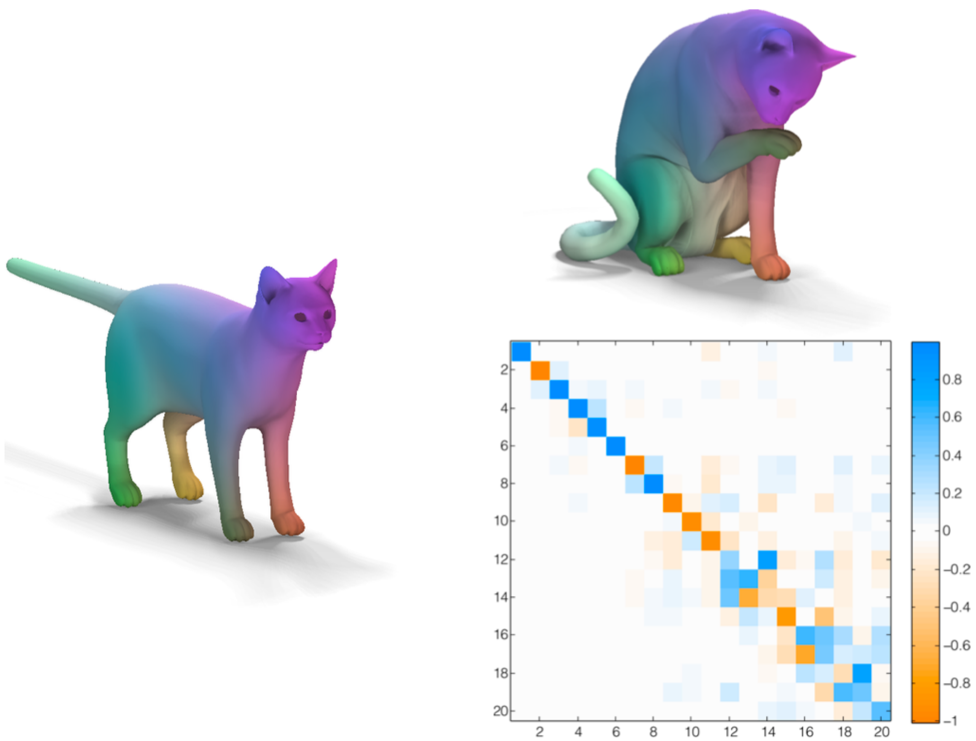
- Operator-based representations.
- Still have some limitations.



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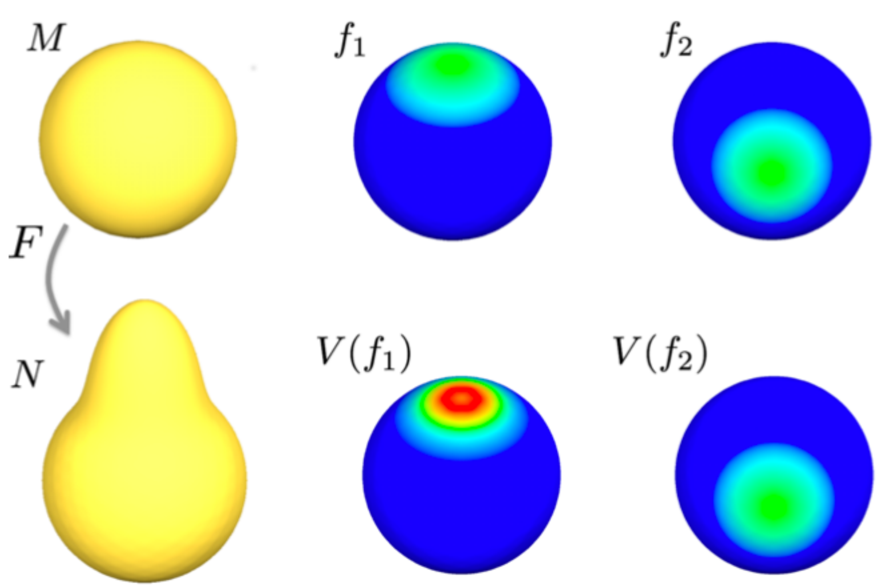


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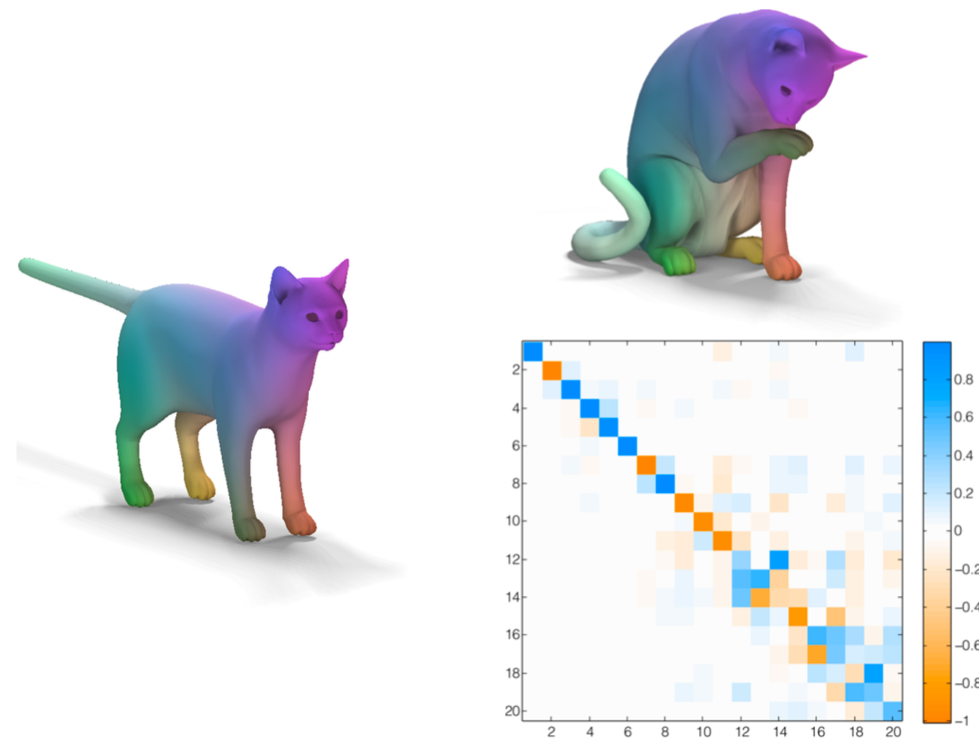
# Motivation and Target

- Operator-based representations.
- Still have some limitations.
- Propose to consider Adjoint Map Representation. Complementary to the existing operators.



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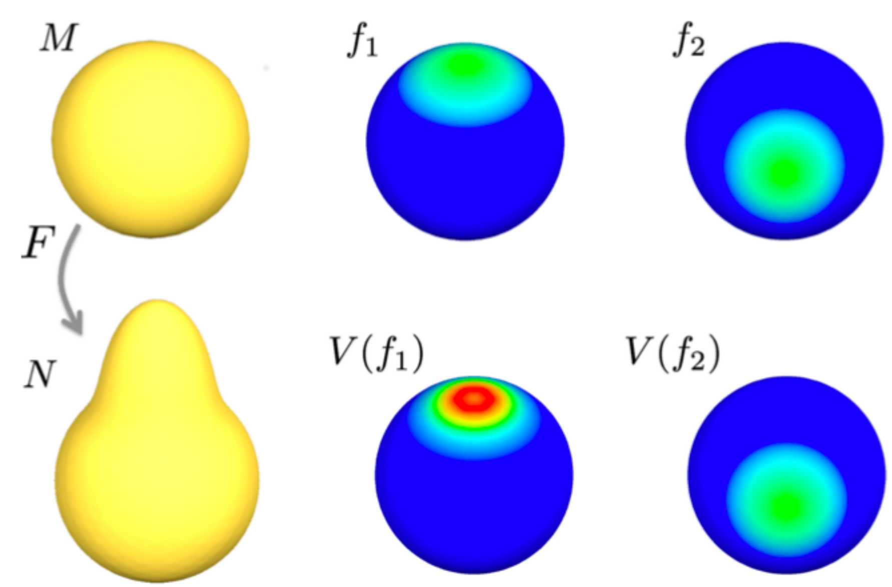


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- **Formulation:** definition, connection to the previous operators, properties.

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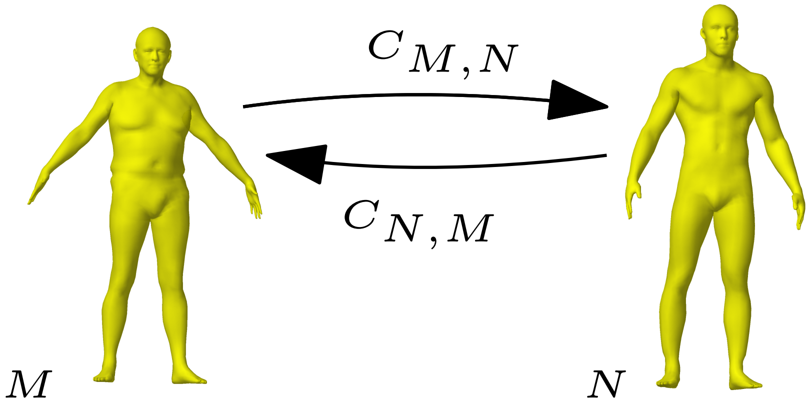
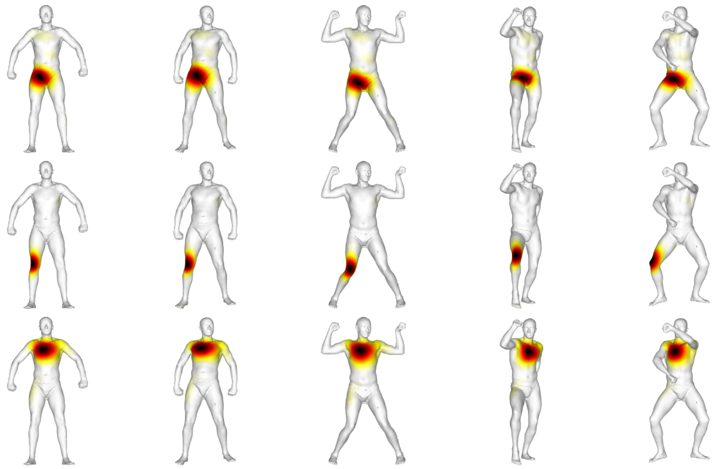
- **Applications:**

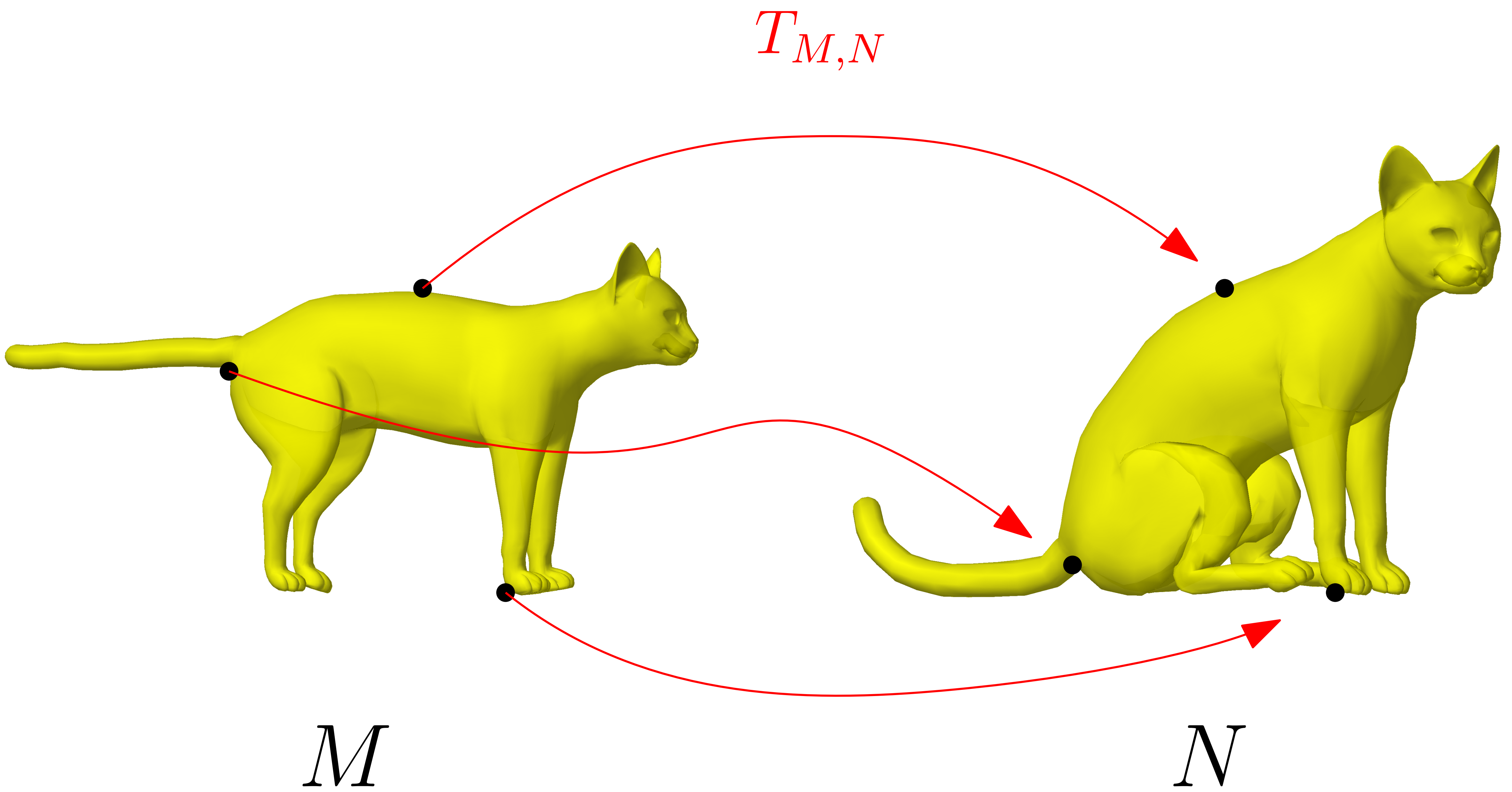



Bi-directional shape matching

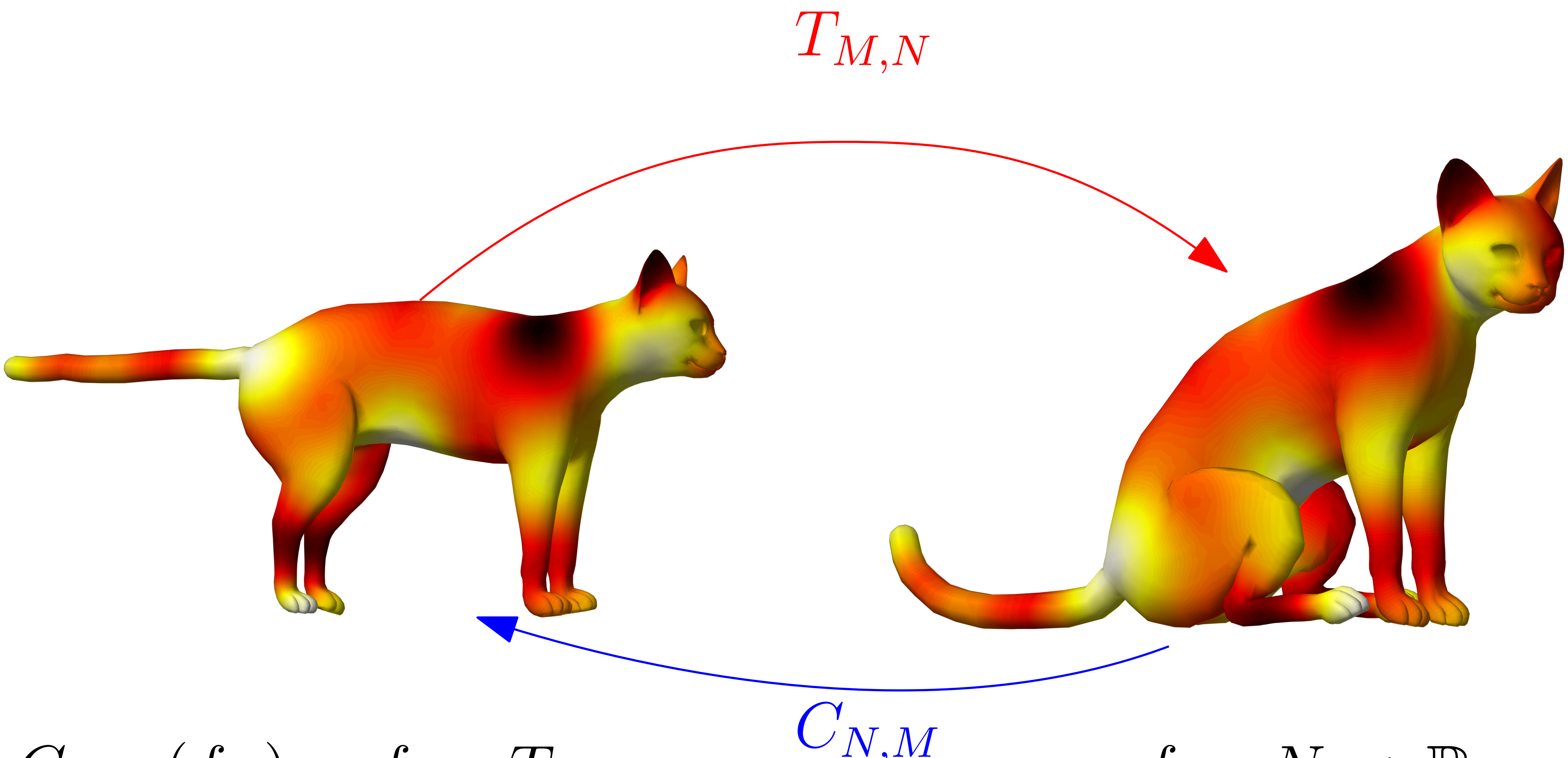


Joint shape Analysis

# Functional Maps



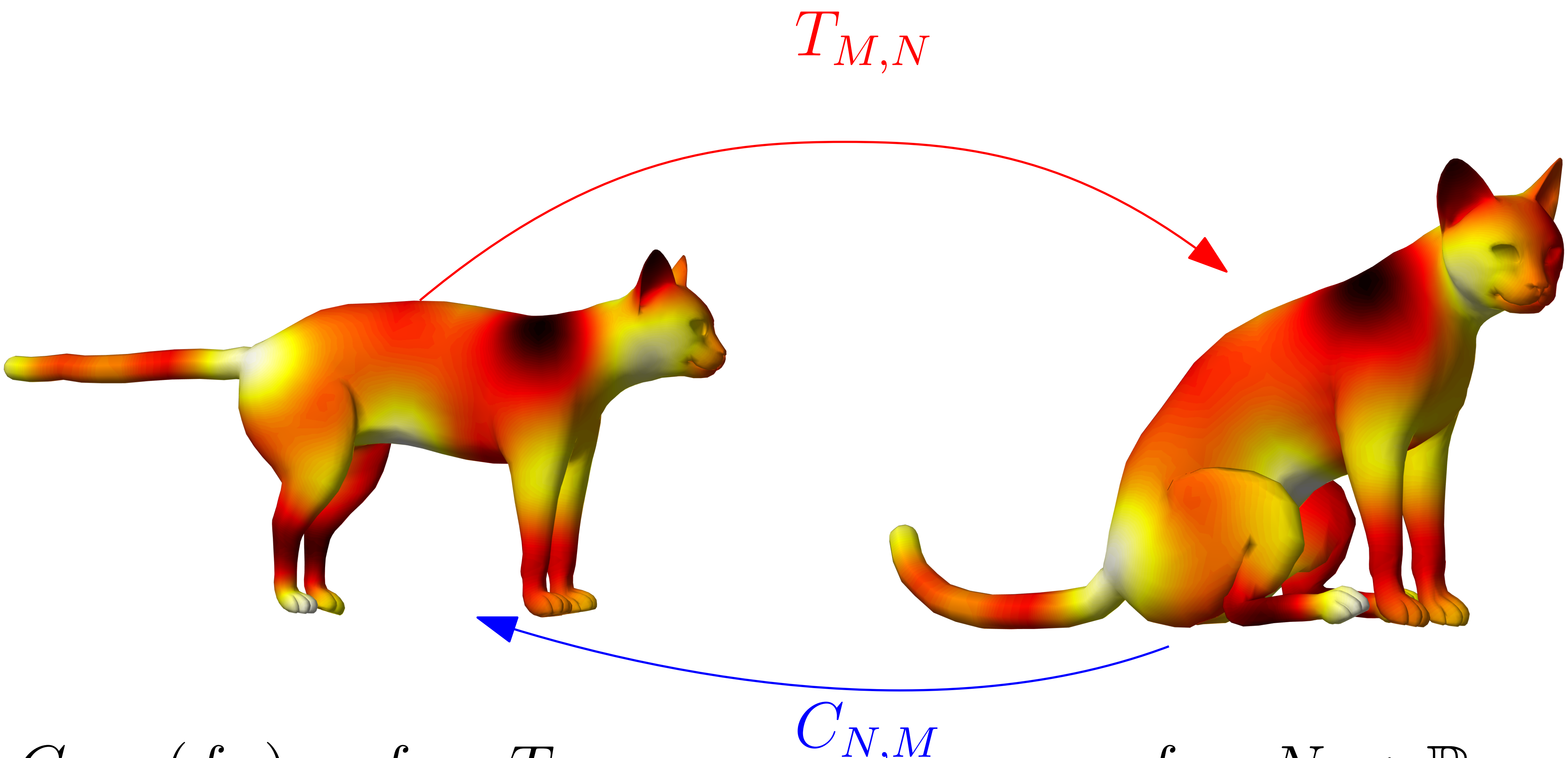
# Functional Maps




$$C_{N,M}(f_N) = f_N \circ T_{M,N}$$

$$f_N : N \to \mathbb{R}$$

# Functional Maps

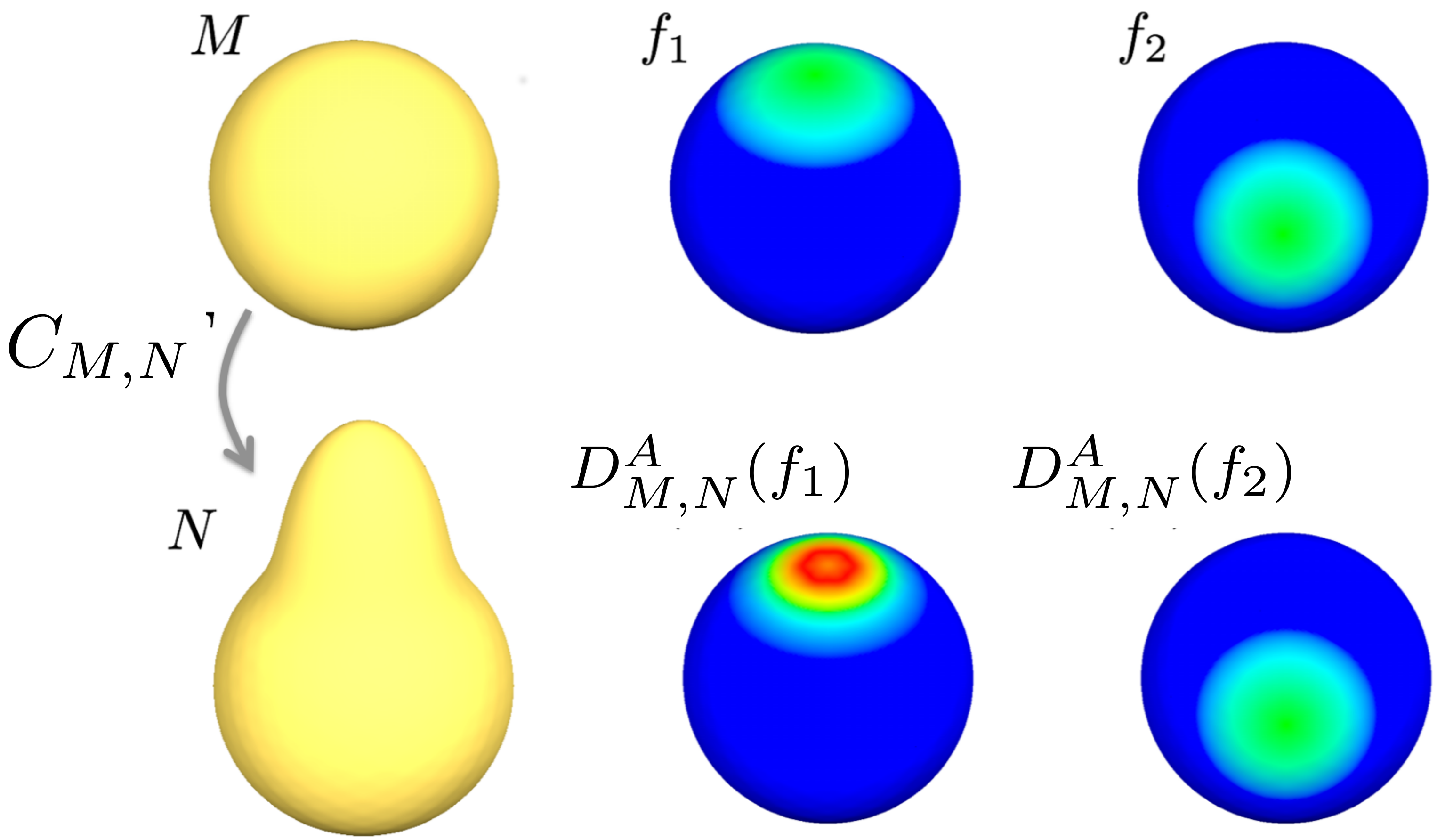



$$C_{N,M}(f_N) = f_N \circ T_{M,N}$$

$$f_N : N \to \mathbb{R}$$

- Preserves function value, but is **not** aware of shape deformation.

# Shape Difference Operators




[R. Rustamov et. al, SIGGRAPH 2013]

$$\int_M g D^A_{M,N}(f) d\nu_M = \int_N C_{M,N}(g) C_{M,N}(f) d\nu_N$$

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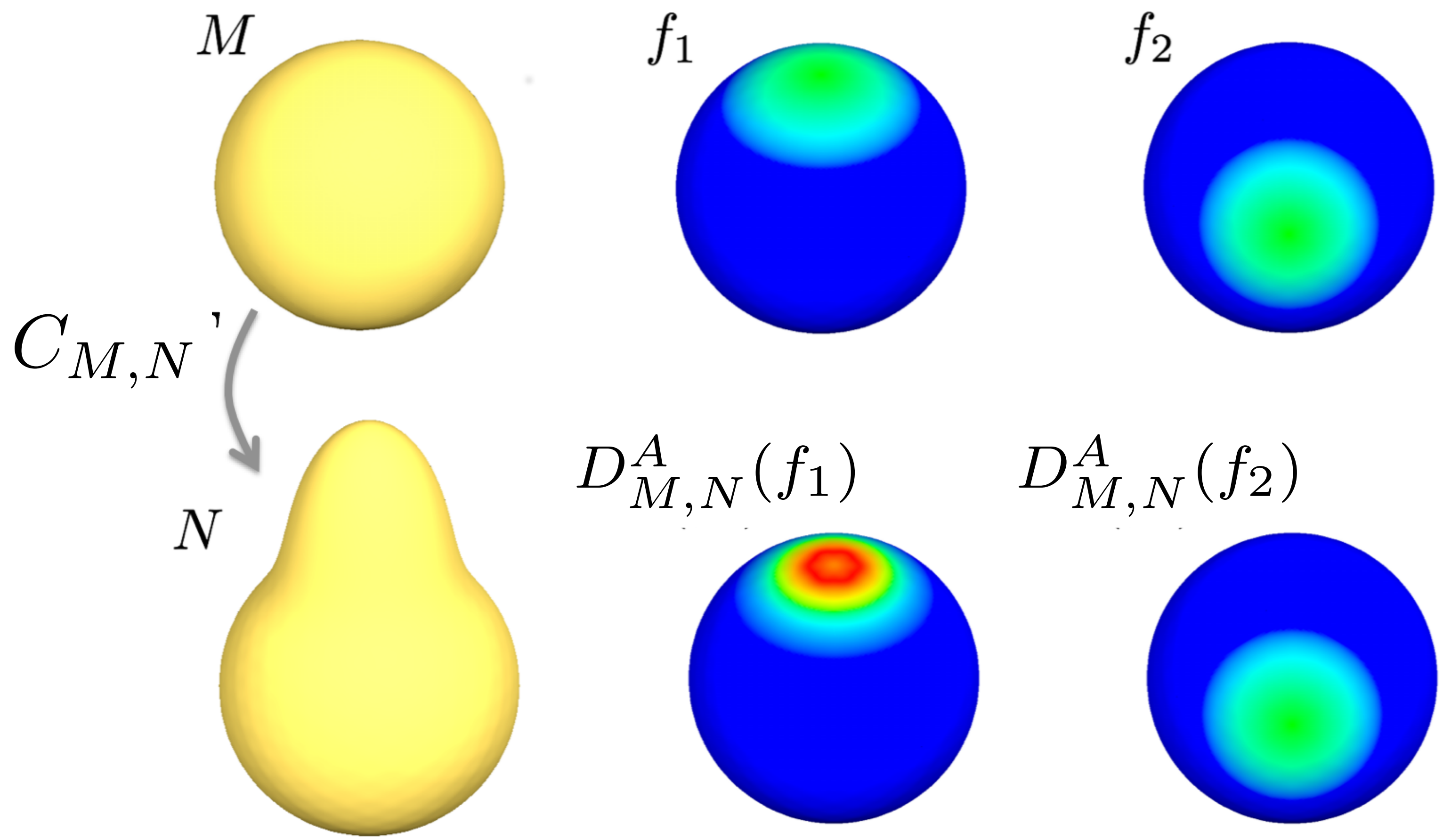


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$$\int_M g D^A_{M,N}(f) d\nu_M = \int_N C_{M,N}(g) C_{M,N}(f) d\nu_N$$

- Does **not** transform information (function) across different shapes.

1 *Map-based Exploration of Intrinsic Shape Differences and Variability*, R. Rustamov et al., SIGGRAPH 2013.

# Formulation

# Adjoint Representation: Definition

**Definition:** Given two shapes $M, N$ and map $T : M \to N$, we define the adjoint functional map, $X_{M,N} : L^2(M) \to L^2(N)$ such that:

$$\langle X_{M,N}(f_M), f_N \rangle_N = \langle f_M, C_{N,M}(f_N) \rangle_M$$

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$L^2-$inner product: $\langle f, g \rangle = \int fg d\nu \Rightarrow X^A_{M,N}$ : Area-based adjoint

$H^1_0-$inner product: $\langle f, g \rangle = \int \nabla f \cdot \nabla g d\nu \Rightarrow X^C_{M,N}$ : Conformal adjoint

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Well-defined

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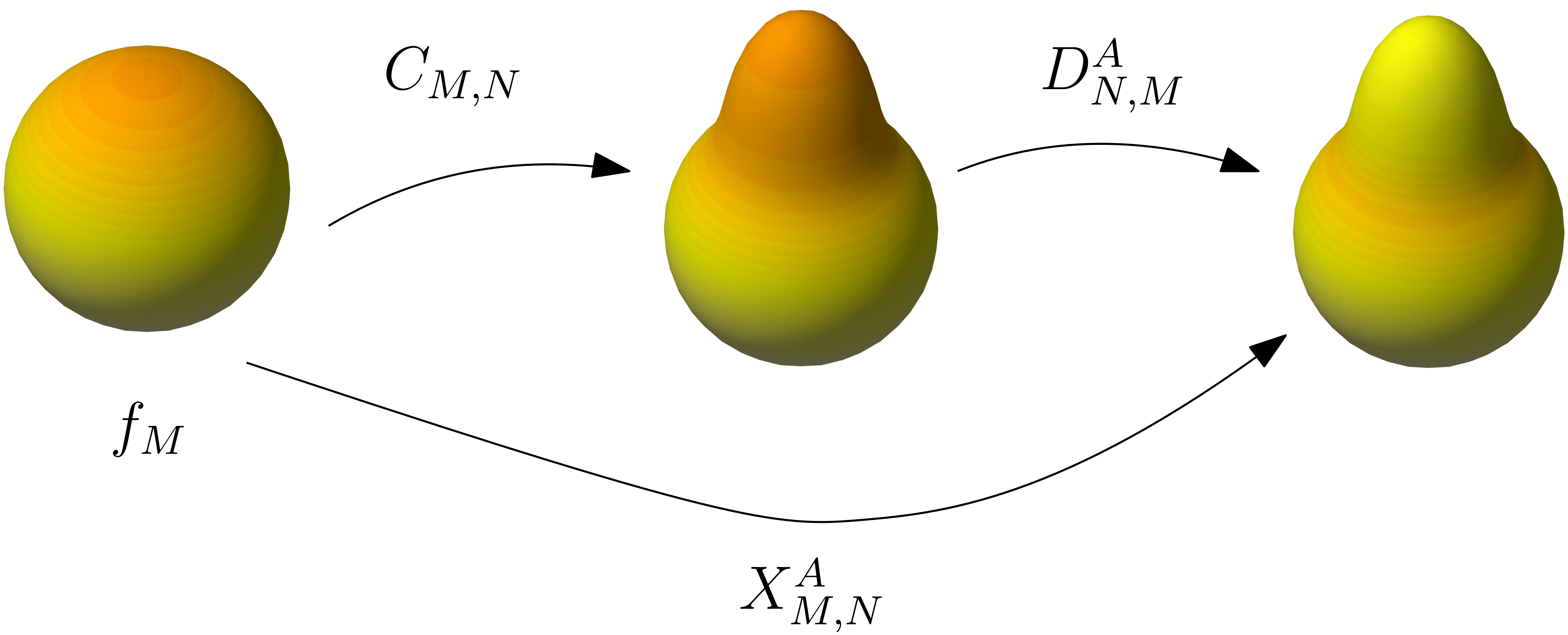
# Adjoint Representation: Property

If $T : M \to N$ is a bijection*, then the induced representations satisfy:

$$X_{M,N} = D_{N,M} C_{M,N}$$



# Adjoint Representation: Property

If $T : M \to N$ is a bijection*:

$X^A_{M,N} = C^{-1}_{N,M}$ if and only if $T$ is area-presearving;

$X^C_{M,N} = C^{-1}_{N,M}$ if and only if $T$ is conformal.

# Adjoint Representation: Discretization

**General Discretization scheme:**

- Shapes are triangulated discrete surfaces
- Low-rank approximation:
  $C_{M,N} : \text{span}\{\phi_1^M, \cdots, \phi_{k_M}^M\} \to \text{span}\{\phi_1^N, \cdots, \phi_{k_N}^N\}$

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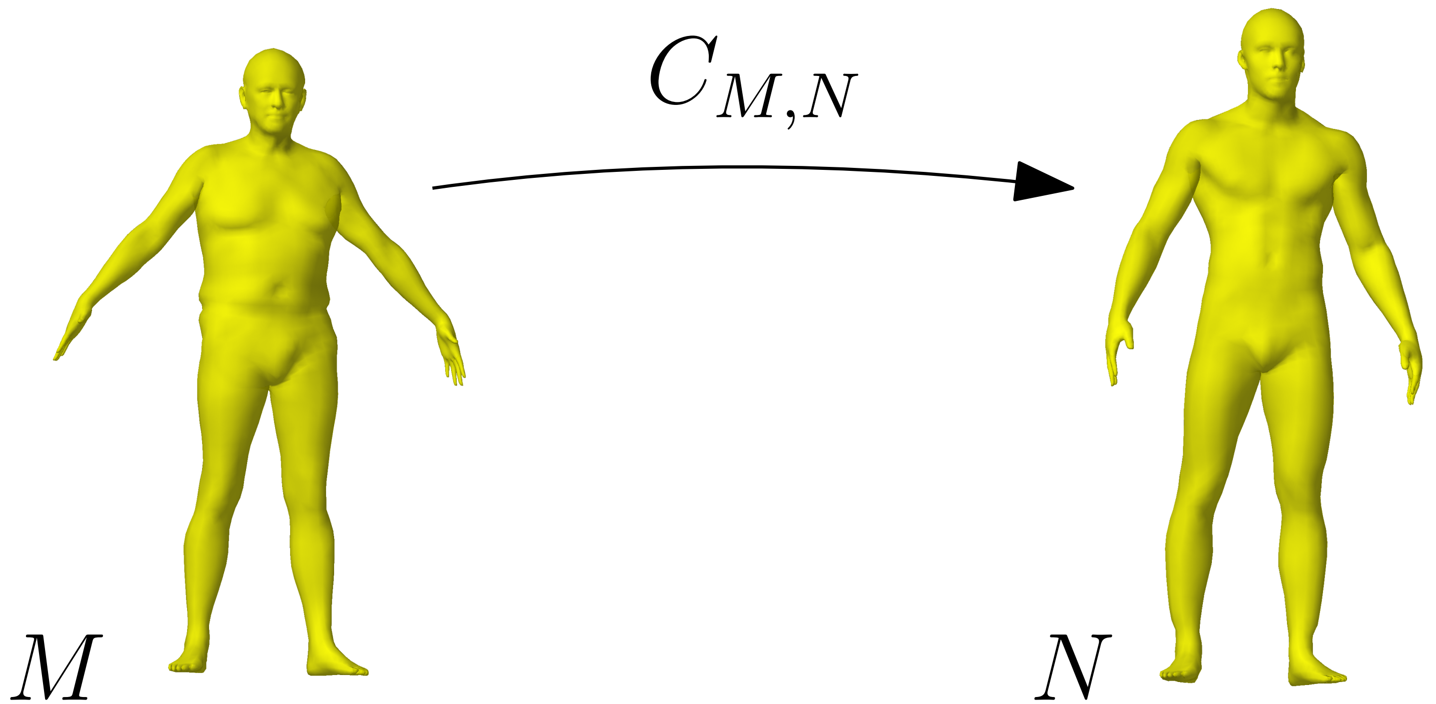
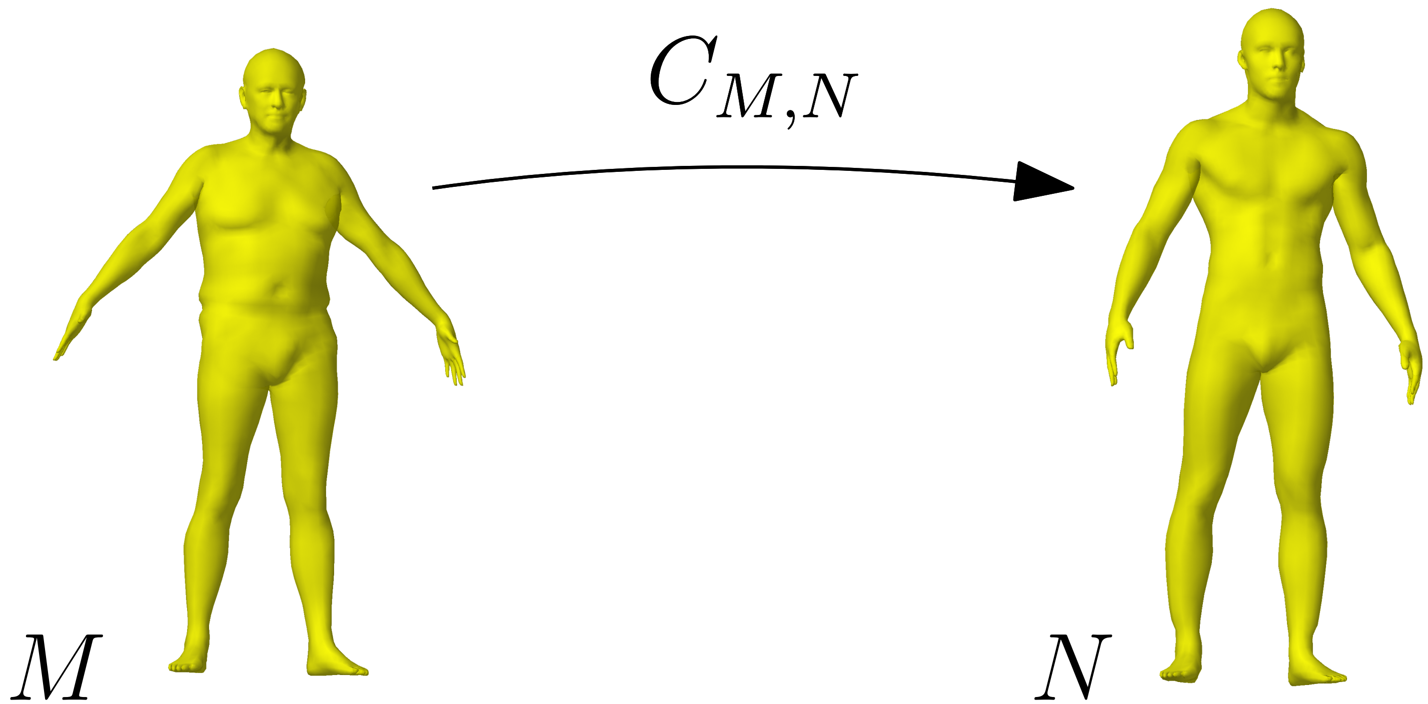
- Area-based: $X_{M,N}^A = C_{N,M}^T$
- Conformal: $X_{M,N}^C = \Lambda_N^+ C_{N,M}^T \Lambda_M$, $\Lambda_M = \text{diag}\{\lambda_1, \cdots, \lambda_{k_M}\}$

# Applications

# Bi-directional Shape Matching



# Bi-directional Shape Matching




$F = [f_1, f_2, \cdots, f_k]$ $\qquad$ $G = [g_1, g_2, \cdots, g_k]$

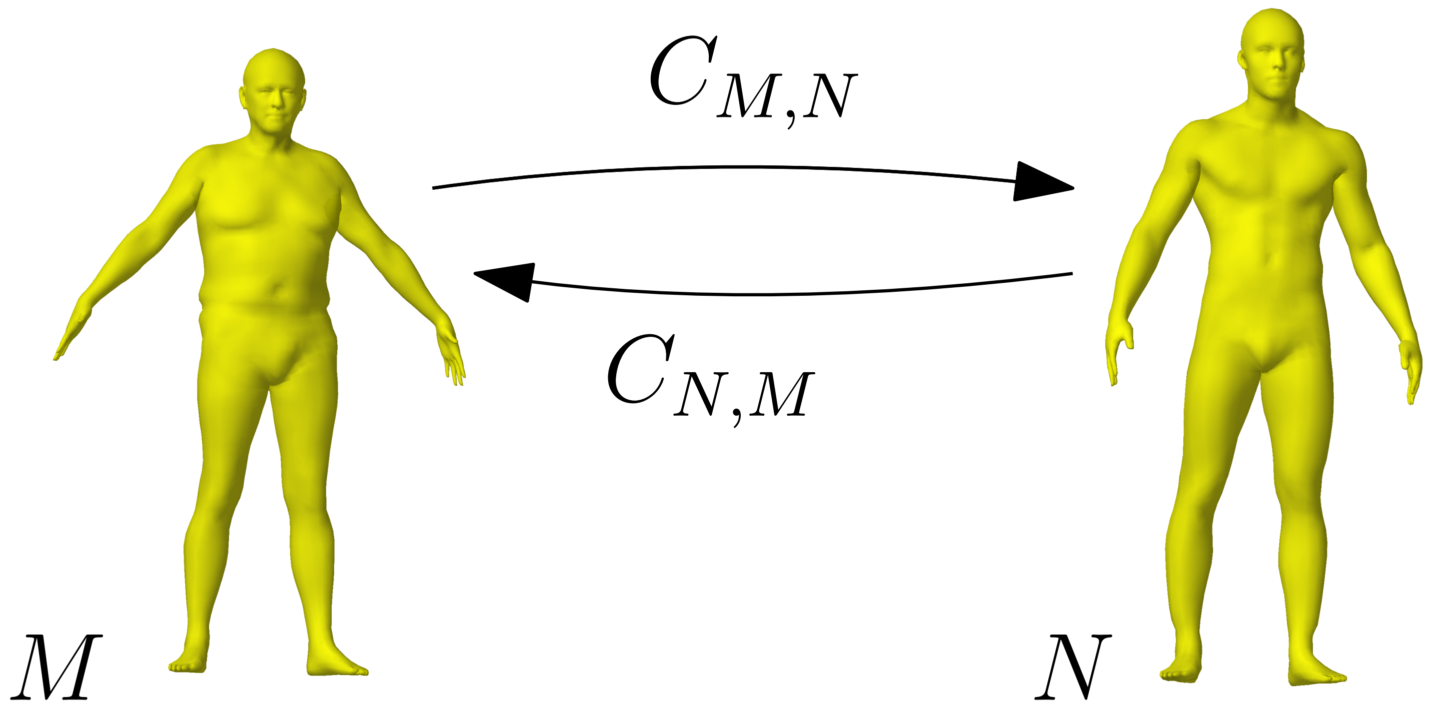
**Standard approach[1] (one-directional)**

$$E_{M,N}(C) = \|CF - G\|^2 + \alpha\|\Delta_N C - C\Delta_M\|^2$$

$$\hat{C}_{M,N} = \arg\min E_{M,N}(C_{M,N})$$

1 *Functional Maps: a Flexible Representation of Maps between Shapes*, M. Ovsjanikov et al., SIGGRAPH 2012.

# Bi-directional Shape Matching

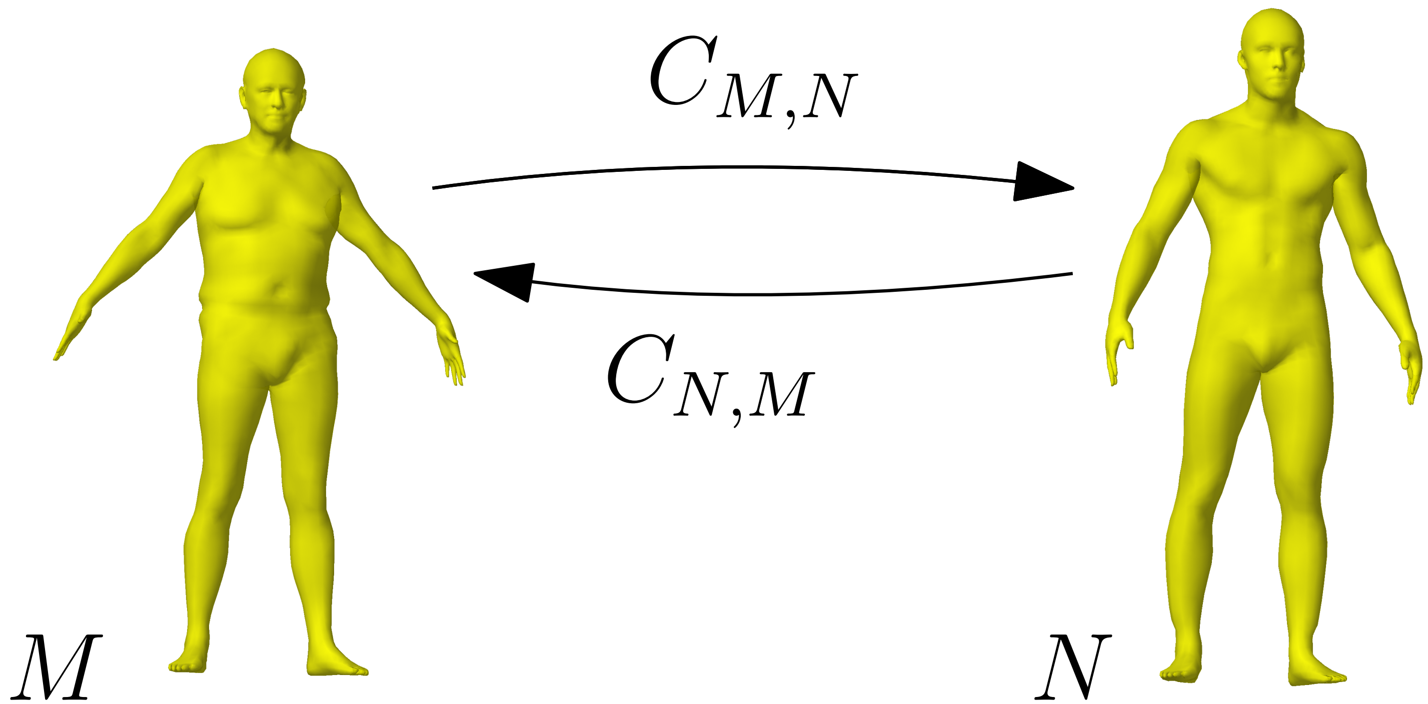



**Coupled Functional Maps**[1]**:**

$$(\hat{C}_{M,N}, \hat{C}_{N,M}) = \text{arg}\min E_{M,N}(C_{M,N}) + E_{N,M}(C_{N,M}) + \beta\|C_{M,N}C_{N,M} - Id\|^2$$

1 *Coupled Functional Maps*, D. Eynard et al., 3DV 2016.

# Bi-directional Shape Matching




**Coupled Functional Maps**[1]**:**

Non-linear term

$$(\hat{C}_{M,N}, \hat{C}_{N,M}) = \arg\min E_{M,N}(C_{M,N}) + E_{N,M}(C_{N,M}) + \beta \|C_{M,N} C_{N,M} - Id\|^2$$

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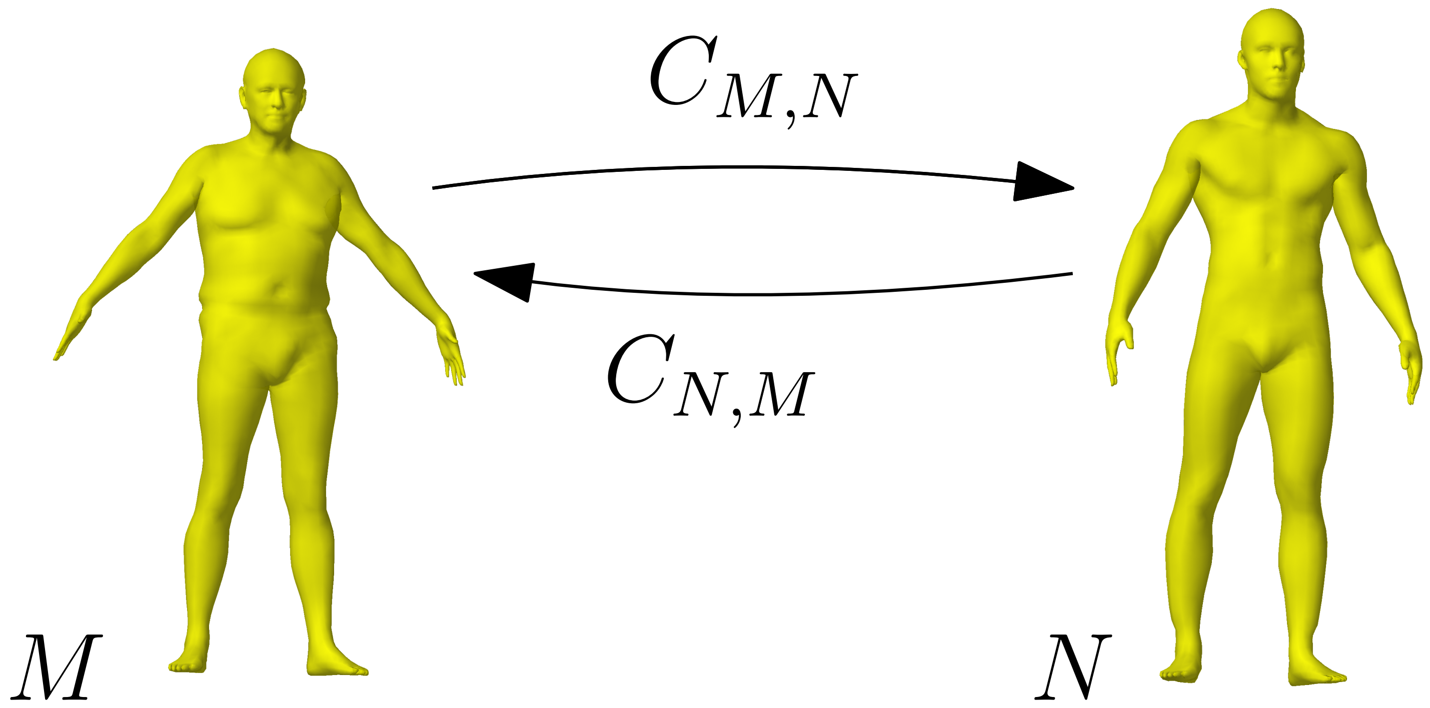
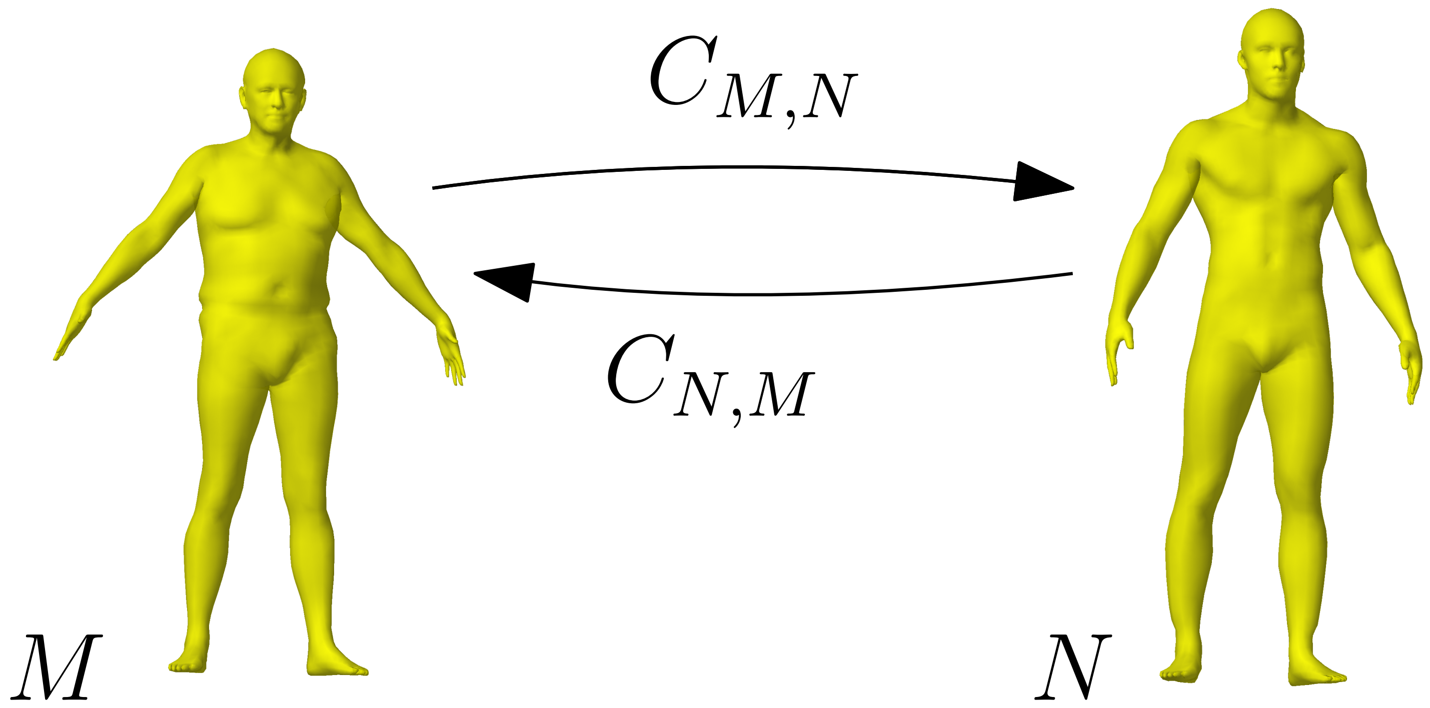



**Our approach:**

$$(\hat{C}_{M,N}, \hat{C}_{N,M}) = \arg\min E_{M,N}(C_{M,N}) + E_{N,M}(C_{N,M}) + \gamma_1 \|X^{A}_{M,N} - C_{M,N}\|^2 + \gamma_2 \|X^{C}_{M,N} - C_{M,N}\|^2$$

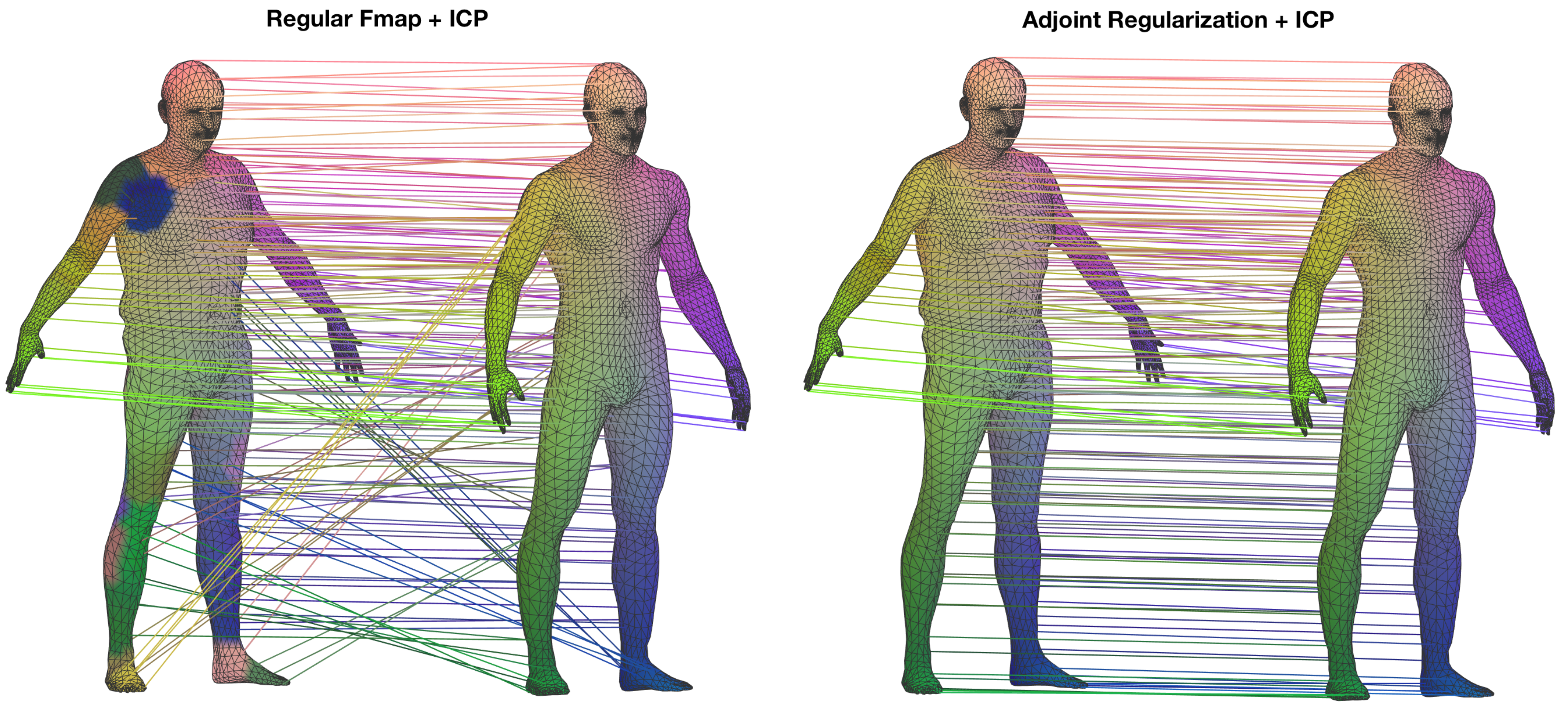
# Bi-directional Shape Matching




**Our approach:**

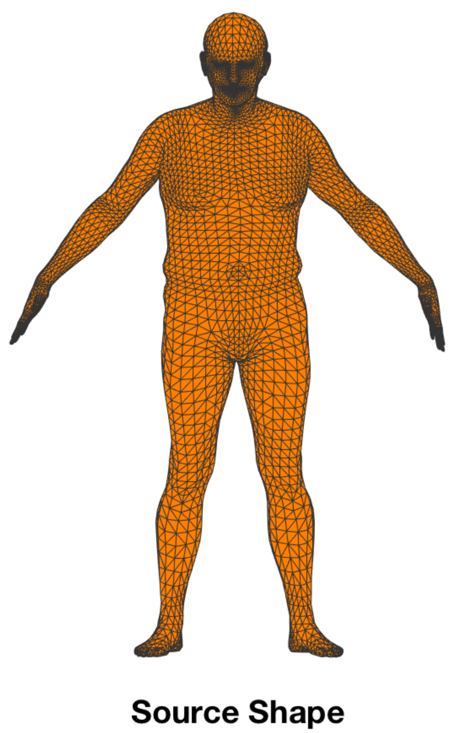
$$
\begin{aligned}
(\hat{C}_{M,N}, \hat{C}_{N,M}) = & \arg\min E_{M,N}(C_{M,N}) + E_{N,M}(C_{N,M}) \\
& + \gamma_1 \|C_{N,M}^T - C_{M,N}\|^2 + \gamma_2 \|\Lambda_N C_{M,N} - C_{N,M}^T \Lambda_M\|^2
\end{aligned}
$$

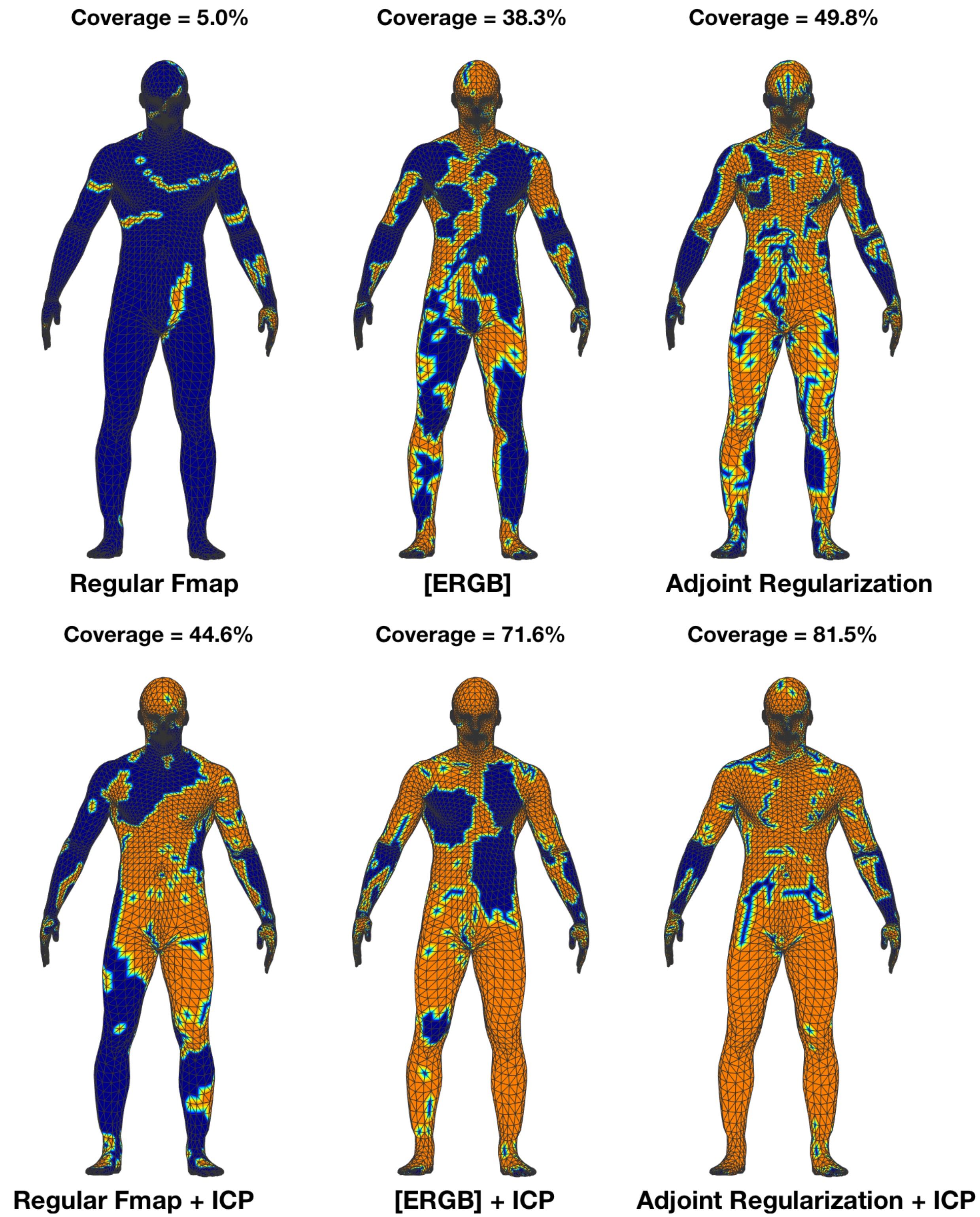
# Bi-directional Shape Matching



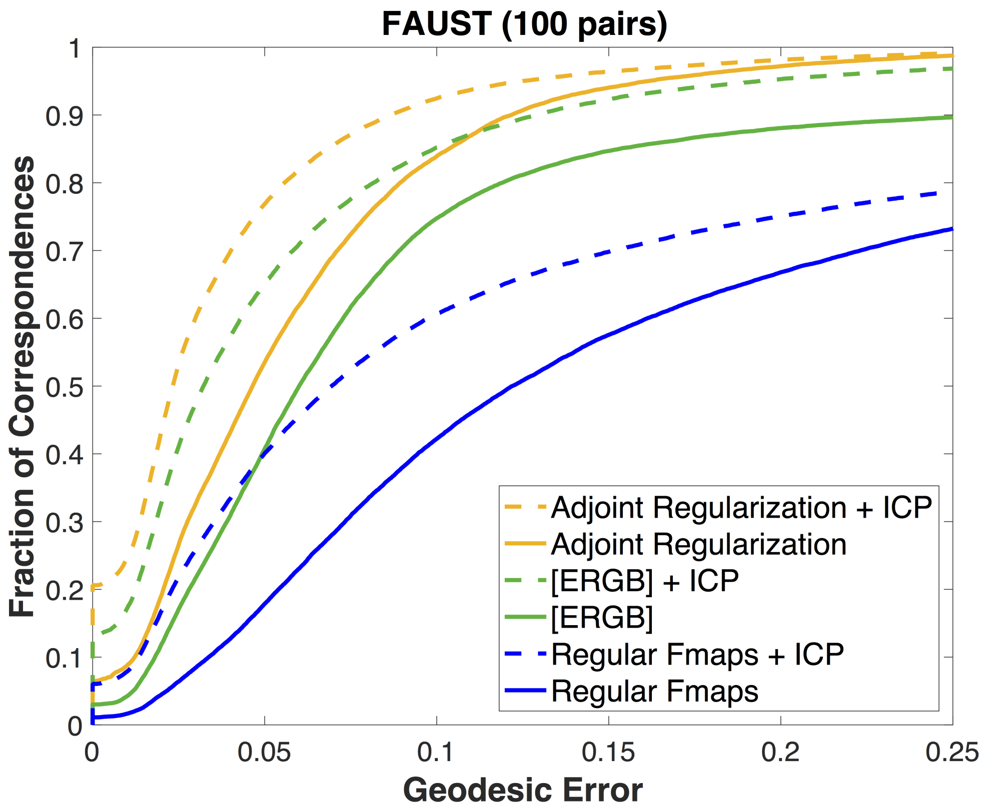
# Bi-directional Shape Matching

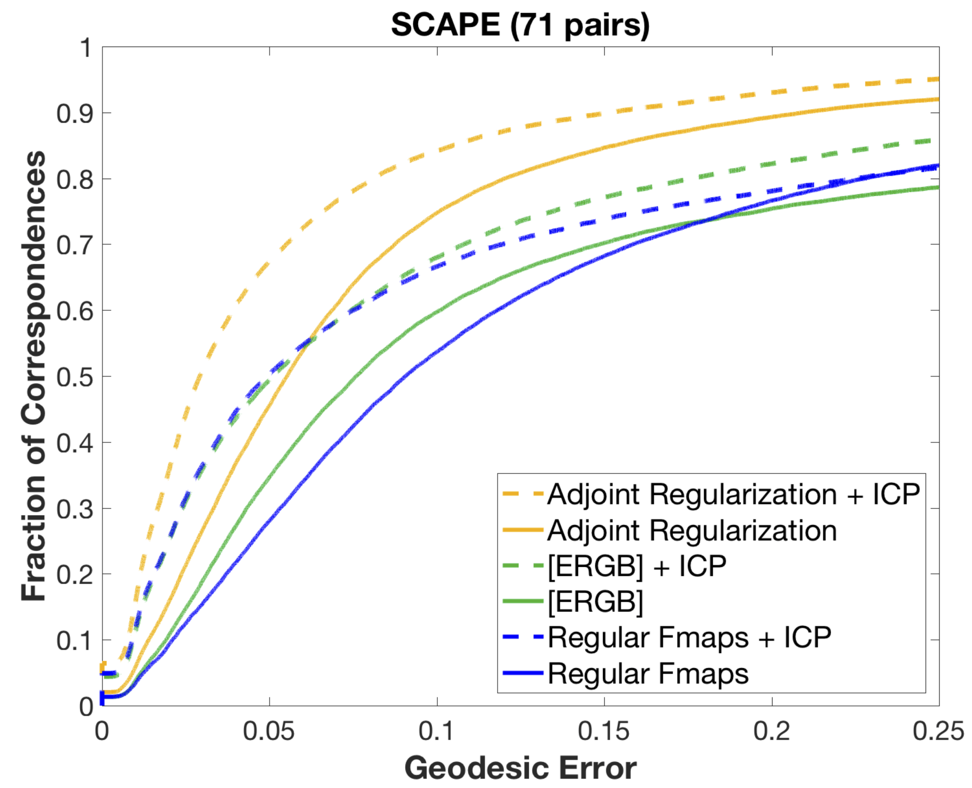
## Invertibility

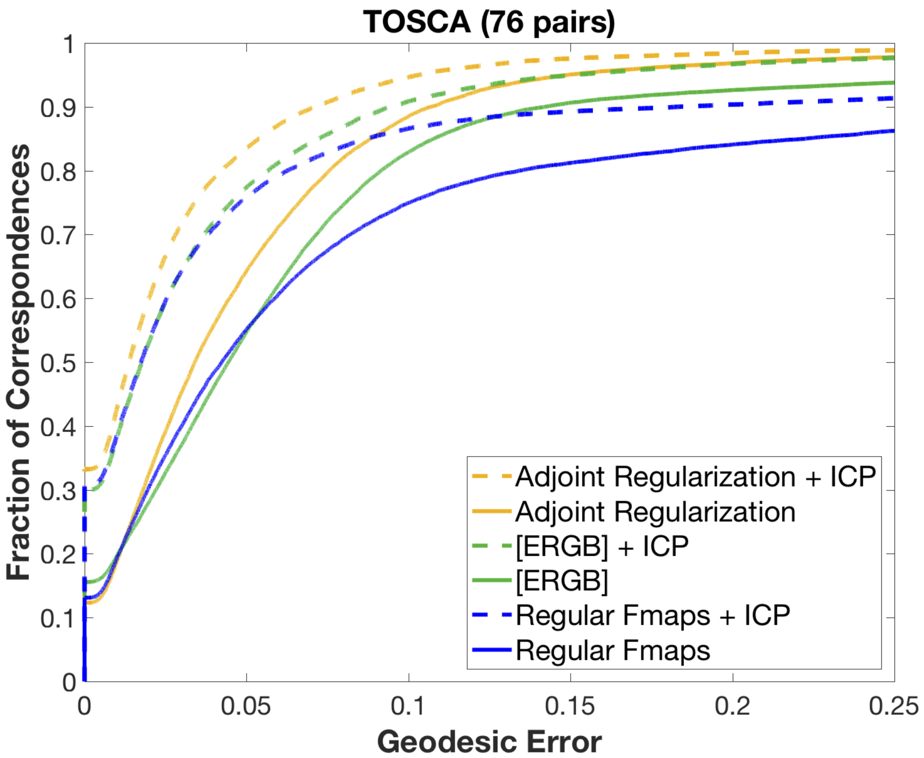


# Bi-directional Shape Matching



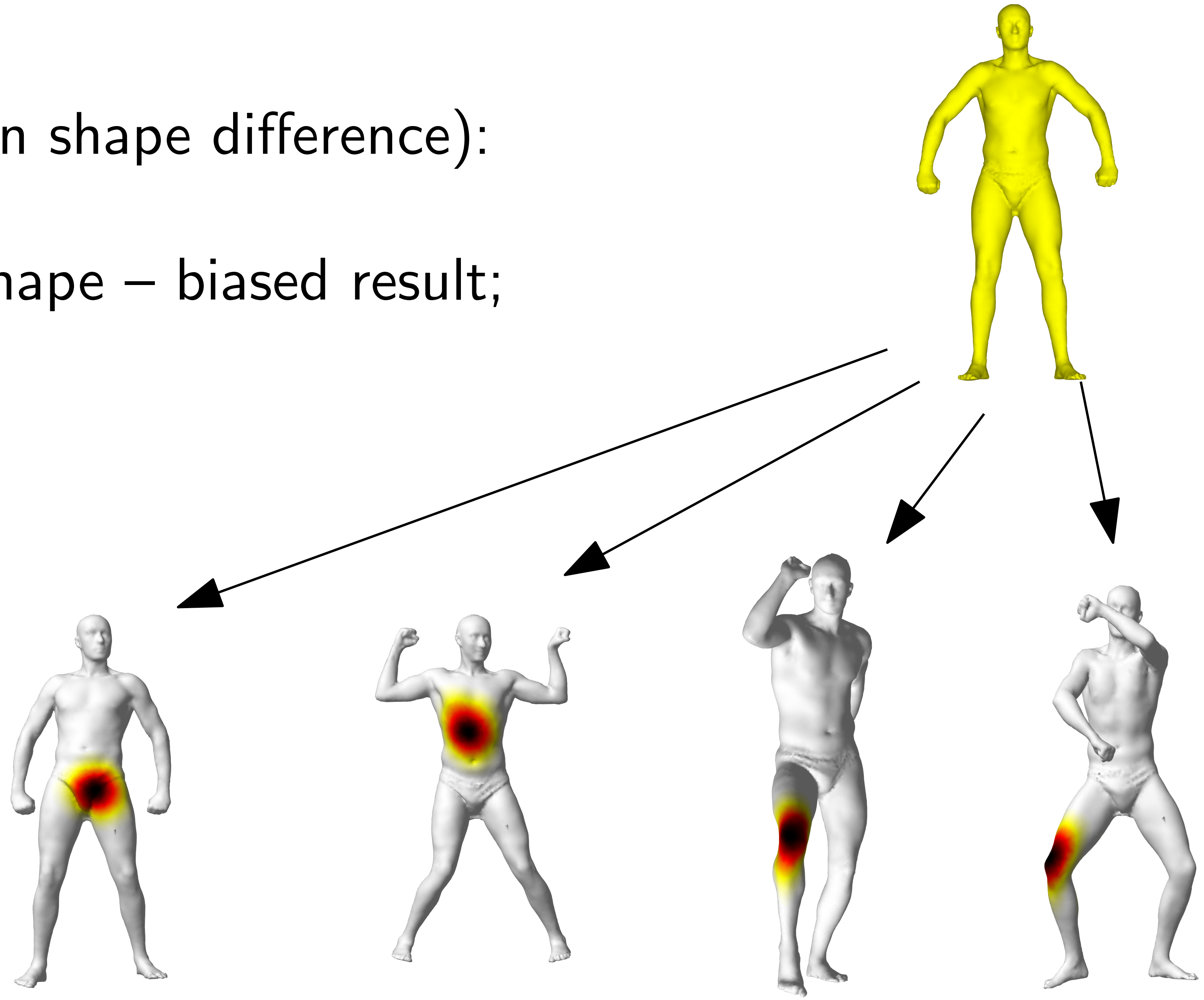
# Joint Shape Analysis

Where are the jointly most distorted areas in the collection?

# Joint Shape Analysis

**Previous approach**[1] (based on shape difference):

- Needs to choose a base shape – biased result;
- Not consistent.



1 *Analysis and Visualization of Maps Between Shapes*, M. Ovsjanikov et al., CGF 2013.

# Joint Shape Analysis

**Our approach** :

- The adjoint functional map reflects shape deformation and transform information across shapes at the same time.
- Extract a collection of consistent basis, using the approach of Wang et al.[1];
- Find the jointly most distorted areas, by using the adjoint maps, rather than the functional maps within the same framework.

1 *Image Co-segmentation via Consistent Functional Maps*, F. Wang et al., ICCV 2013.

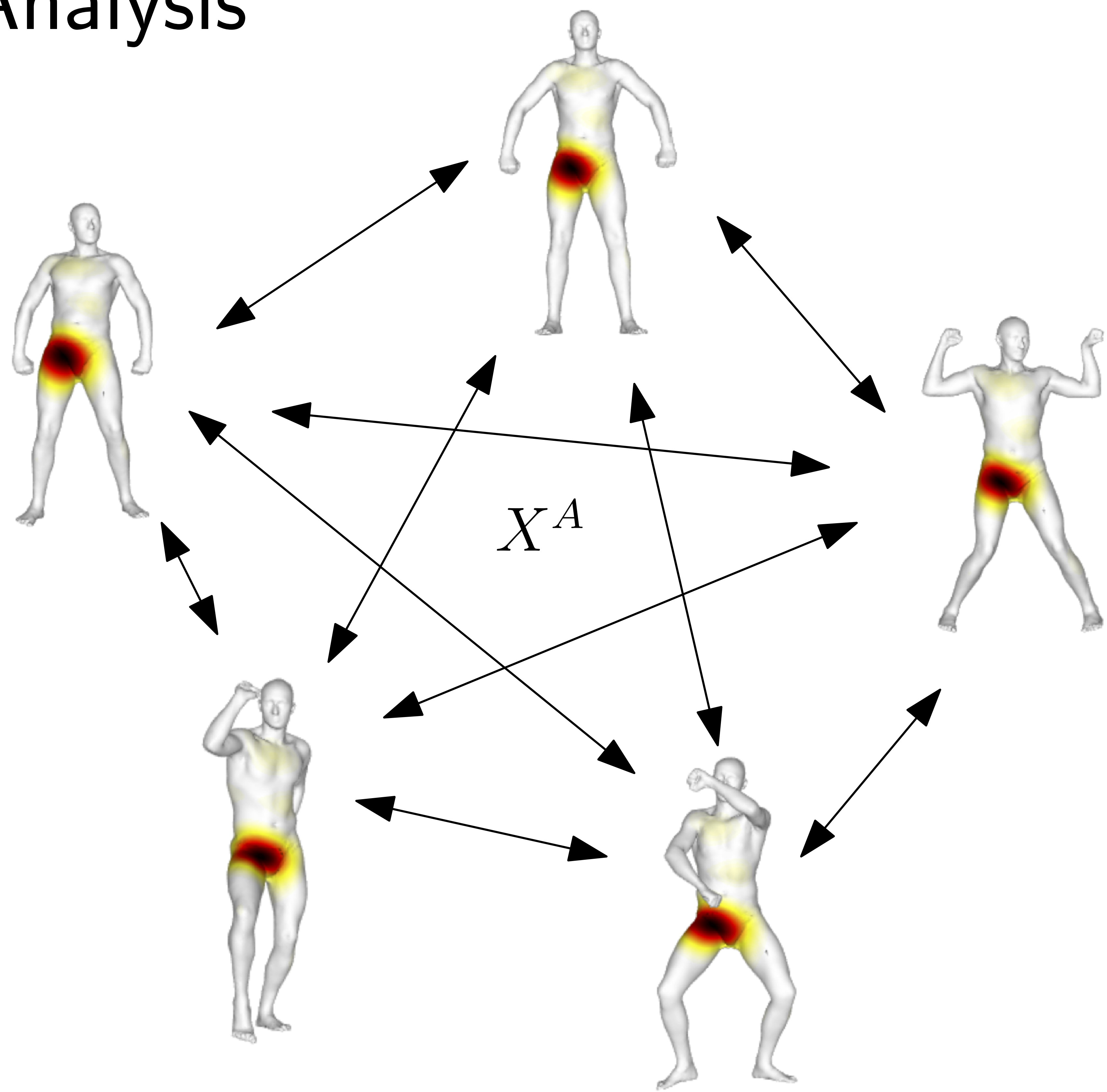
# Joint Shape Analysis

**Our approach** :

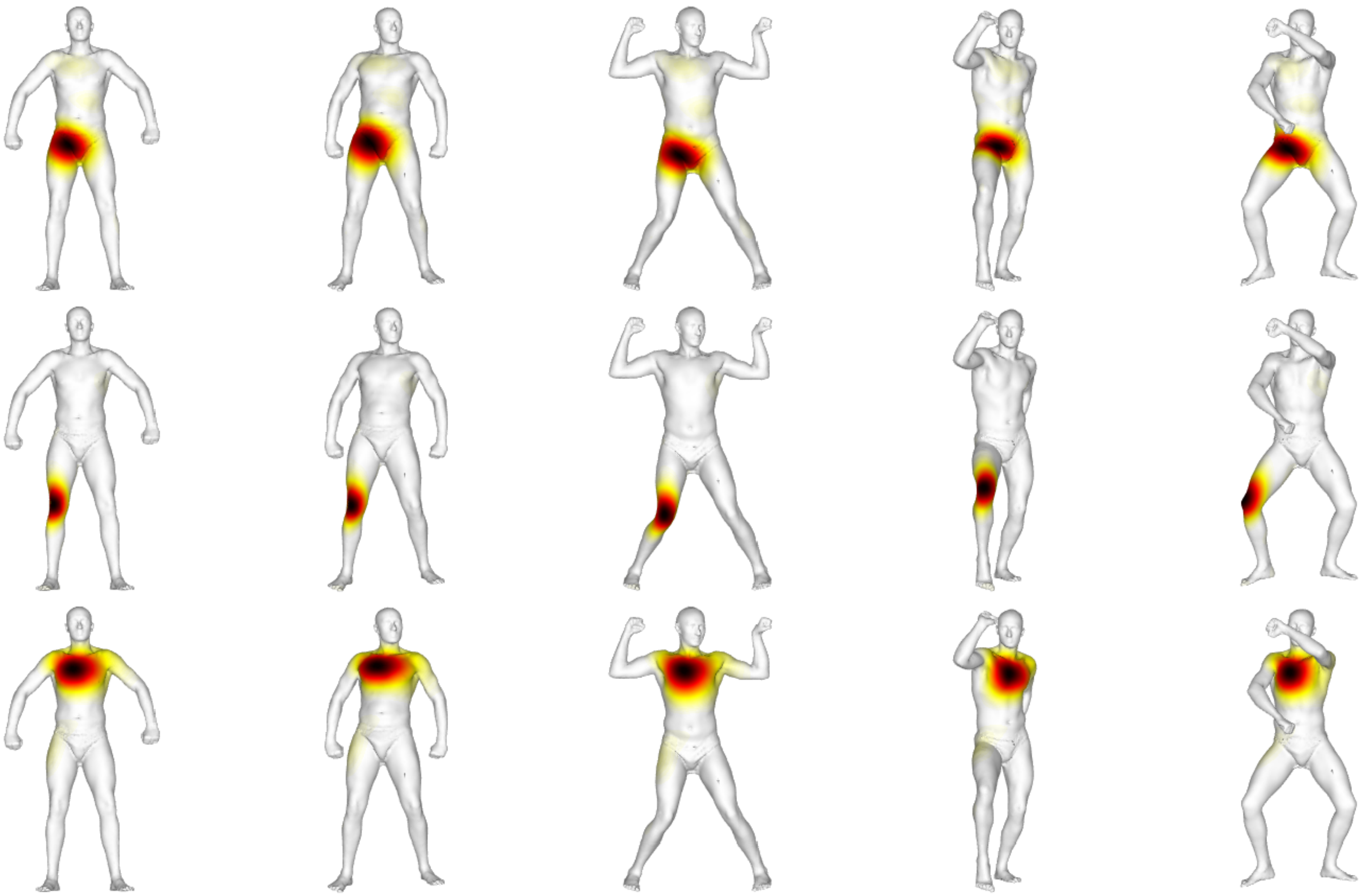
- Find consistent $f = [f_1, f_2, ..., f_n]$ s.t. $\sum_{i,j} \|C_{ij} f_i - f_j\|_{L^2} = 0$
- In the space spanned by all the consistent $f$, maximize $\sum_{i,j} \|X_{ij} f_i - f_j\|_{L^2}$.

# Joint Shape Analysis

**Our approach** :

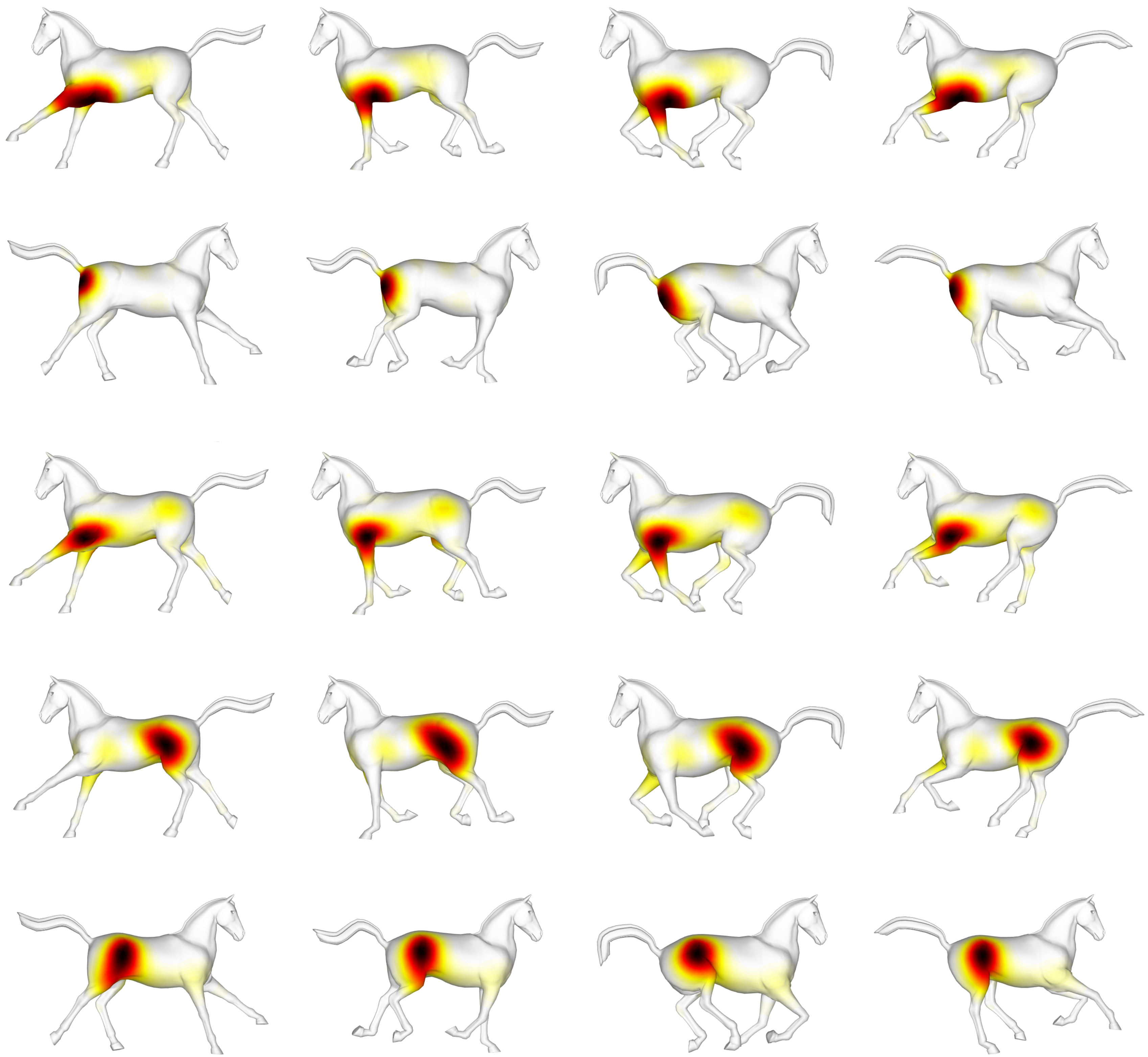


# Joint Shape Analysis

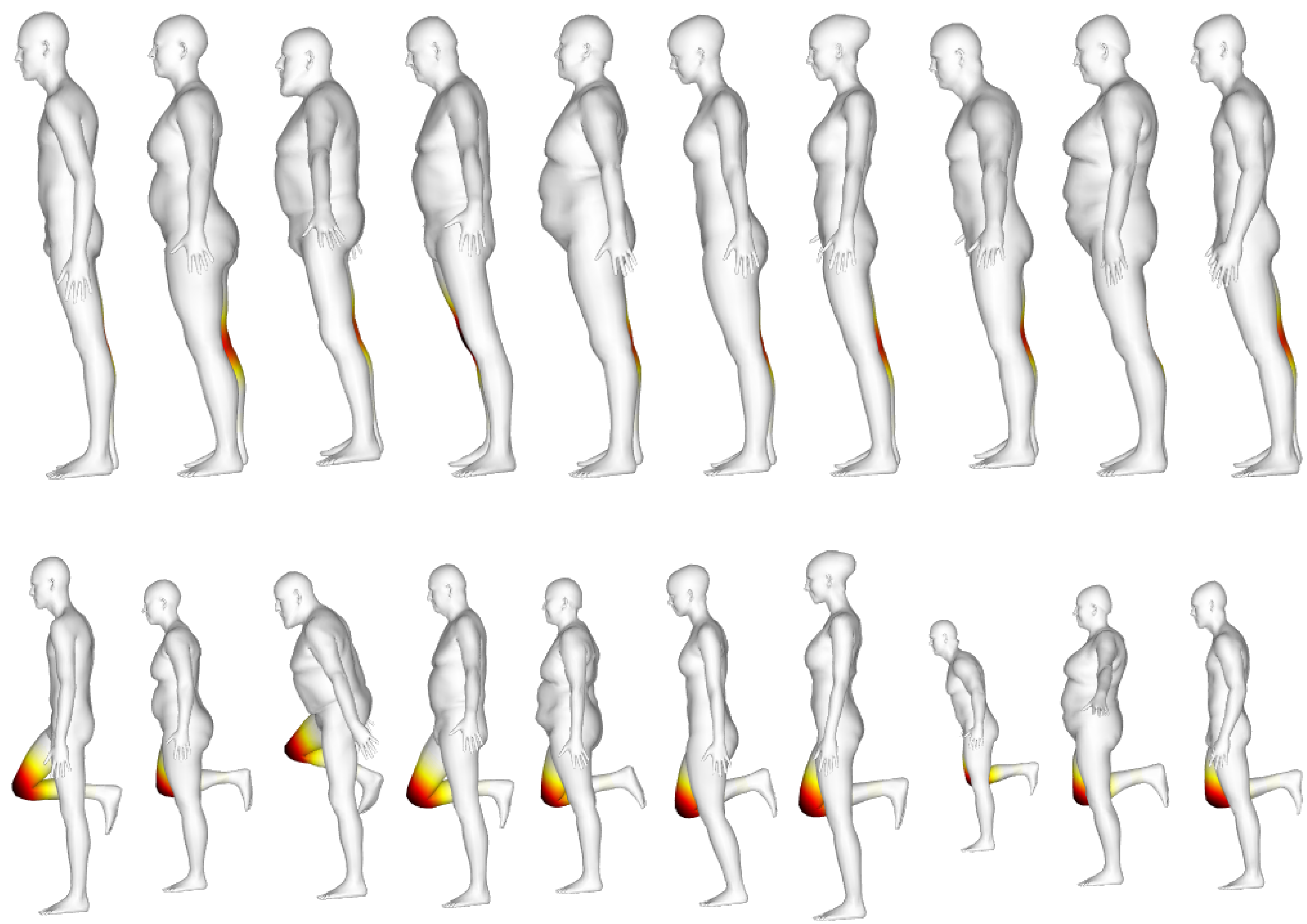


Order the highlighted deformations with a unified measure.

# Joint Shape Analysis



# Something New Ongoing



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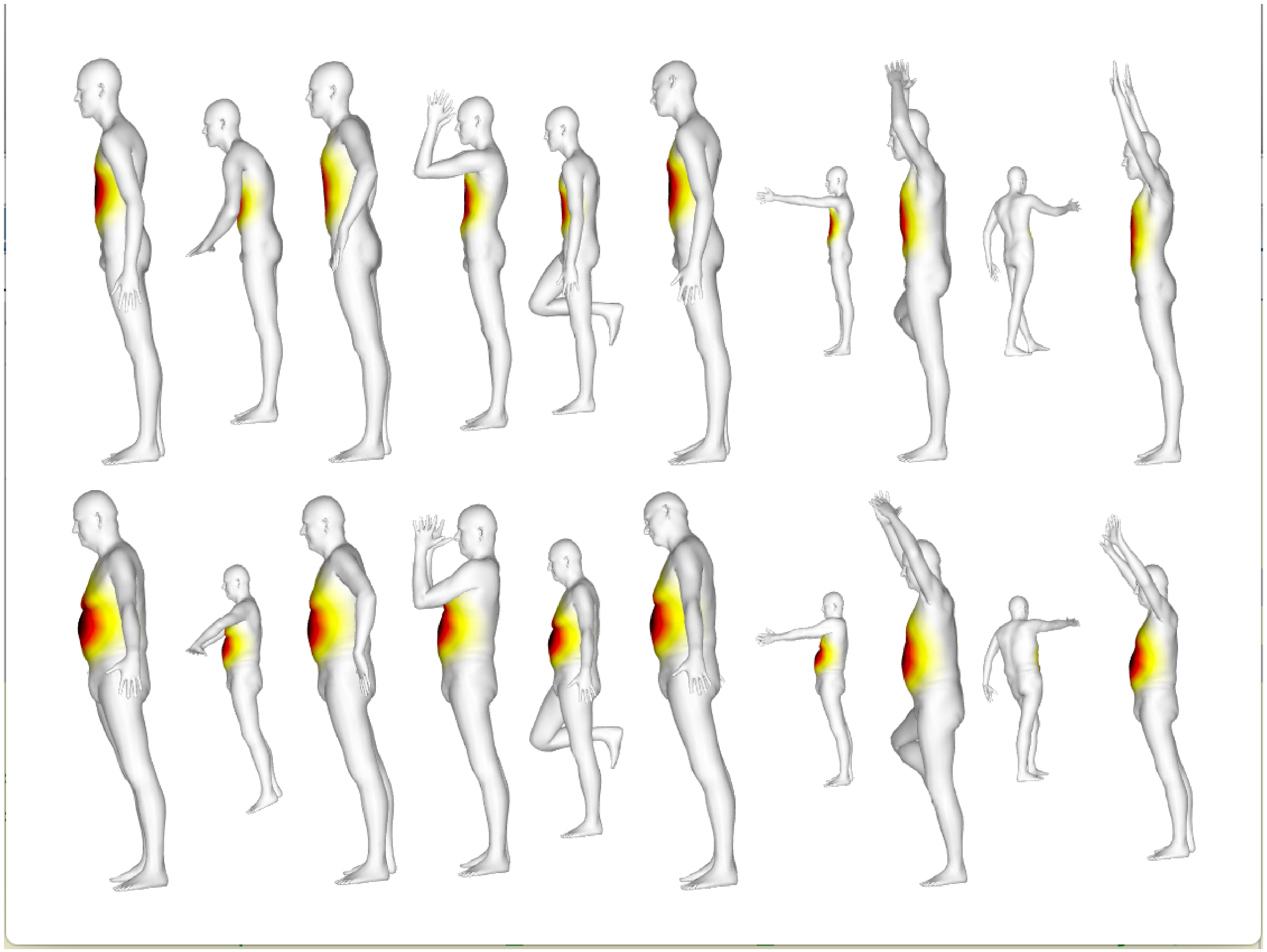

# Conclusion

The adjoint map representation

- both transfers information and reflects deformation.
- can be obtained without extra effort.

# Thanks for your attention! Questions?

Acknowledgement: This work is supported by Marie-Curie CIG-334283, a CNRS chaire d'excellence, chaire Jean Marjoulet from Ecole Polytechnique, FUI project TANDEM 2, and a Google Focused Research Award.

# Backup

**Proposition.** Let $X^A_{M,N}$ be the area-based adjoint operator of $C_{N,M}$, and let $\kappa = X^A_{N,M}(\mathbf{1}_N)$, then we have

$$X^A_{M,N}(\kappa \cdot f) = C_{M,N}(f)$$