Adjoint Map Representation for Shape Analysis and Matching

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Joint work with Maks Ovsjanikov.
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Motivation and Target

- Operator-based representations.

1. Functional Maps: a Flexible Representation of Maps between Shapes, M. Ovsjanikov et al., SIGGRAPH, 2012;
Motivation and Target

- Operator-based representations.
- Still have some limitations.

Functional maps$^1$

Shape difference$^2$

Motivation and Target

- Operator-based representations.
- Still have some limitations.
- Propose to consider Adjoint Map Representation. Complementary to the existing operators.

1 Functional Maps: a Flexible Representation of Maps between Shapes, M. Ovsjanikov et al., SIGGRAPH, 2012;
2 Map-based Exploration of Intrinsic Shape Differences and Variability, R. Rustamov et al., SIGGRAPH 2013.
Overview
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- **Formulation**: definition, connection to the previous operators, properties.
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- **Applications**:

  ![Bi-directional shape matching](image)
  ![Joint shape Analysis](image)

  Bi-directional shape matching    Joint shape Analysis
Functional Maps

$T_{M,N}$

$M$  \hspace{2cm} $N$
Functional Maps

\[ f_N : N \rightarrow \mathbb{R} \]

\[ C_{N,M}(f_N) = f_N \circ T_{M,N} \]

\[ f_N : N \rightarrow \mathbb{R} \]
**Functional Maps**

\[ f_N : N \rightarrow \mathbb{R} \]

\[ C_{N,M}(f_N) = f_N \circ T_{M,N} \]

- Preserves function value, but is **not** aware of shape deformation.
Shape Difference Operators

\[
\int_M g \cdot D_{M,N}^A(f) \, d\nu_M = \int_N C_{M,N}(g) \cdot C_{M,N}(f) \, d\nu_N
\]

[R. Rustamov et al., SIGGRAPH 2013]

1 Map-based Exploration of Intrinsic Shape Differences and Variability, R. Rustamov et al., SIGGRAPH 2013.
Shape Difference Operators

\[ \int_M gD_{M,N}^A(f) d\nu_M = \int_N C_{M,N}(g)C_{M,N}(f) d\nu_N \]

- Does not transform information (function) across different shapes.

1 Map-based Exploration of Intrinsic Shape Differences and Variability, R. Rustamov et al., SIGGRAPH 2013.
Formulation
**Definition:** Given two shapes $M, N$ and map $T : M \to N$, we define the adjoint functional map, $X_{M,N} : L^2(M) \to L^2(N)$ such that:

$$\langle X_{M,N}(f_M), f_N \rangle_N = \langle f_M, C_{N,M}(f_N) \rangle_M$$
Adjoint Representation: Definition

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$L^2$–inner product: $\langle f, g \rangle = \int fg d\nu \Rightarrow X_{M,N}^A : \text{Area-based adjoint}$

$H^1_0$–inner product: $\langle f, g \rangle = \int \nabla f \cdot \nabla g d\nu \Rightarrow X_{M,N}^C : \text{Conformal adjoint}$
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\[
\langle X_{M,N}(f_M), f_N \rangle_N = \langle f_M, f_N \circ T \rangle_M
\]

\( L^2 \)–inner product: \( \langle f, g \rangle = \int fg \, dv \) \( \Rightarrow X_{M,N}^A : \) Area-based adjoint

\( H^1_0 \)–inner product: \( \langle f, g \rangle = \int \nabla f \cdot \nabla g \, dv \) \( \Rightarrow X_{M,N}^C : \) Conformal adjoint

Well-defined
Adjoint Representation: Property

If \( T : M \to N \) is a bijection*, then the induced representations satisfy:

\[
X_{M,N} = D_{N,M} C_{M,N}
\]

\( f_M \)

\( C_{M,N} \)

\( D_{N,M}^A \)

\( X_{M,N}^A \)
Adjoint Representation: Property

If $T : M \to N$ is a bijection*:

\[ X^A_{M,N} = C^{-1}_{N,M} \text{ if and only if } T \text{ is area-preserving;} \]

\[ X^C_{M,N} = C^{-1}_{N,M} \text{ if and only if } T \text{ is conformal.} \]
General Discretization scheme:

- Shapes are triangulated discrete surfaces
- Low-rank approximation:
  \[ C_{M,N} : \text{span}\{\phi^M_1, \cdots, \phi^M_{k_M}\} \rightarrow \text{span}\{\phi^N_1, \cdots, \phi^N_{k_N}\} \]
Adjoint Representation: Discretization

General Discretization scheme:

- Shapes are triangulated discrete surfaces
- Low-rank approximation:
  \[ C_{M,N} : \text{span}\{\phi_1^M, \cdots, \phi_{k_M}^M\} \rightarrow \text{span}\{\phi_1^N, \cdots, \phi_{k_N}^N\} \]
- Area-based: \( X_{M,N}^A = C_{N,M}^T \)
- Conformal: \( X_{M,N}^C = \Lambda_+^N C_{N,M}^T \Lambda_M, \Lambda_M = \text{diag}\{\lambda_1, \cdots, \lambda_{k_M}\} \)
Applications
Bi-directional Shape Matching

\[ C_{M,N} \]
Bi-directional Shape Matching

Standard approach\(^1\) (one-directional)

\[
E_{M,N}(C) = \|CF - G\|^2 + \alpha\|\Delta_NC - C\Delta_M\|^2
\]

\[
\hat{C}_{M,N} = \arg \min E_{M,N}(C_{M,N})
\]

\(^1\) Functional Maps: a Flexible Representation of Maps between Shapes, M. Ovsjanikov et al., SIGGRAPH 2012.
Bi-directional Shape Matching

Coupled Functional Maps\textsuperscript{1}:

\[
(\hat{C}_{M,N}, \hat{C}_{N,M}) = \arg\min E_{M,N}(C_{M,N}) + E_{N,M}(C_{N,M}) \\
+ \beta \|C_{M,N}C_{N,M} - Id\|^2
\]

\textsuperscript{1} Coupled Functional Maps, D. Eynard et al., 3DV 2016.
Bi-directional Shape Matching

Coupled Functional Maps\textsuperscript{1}:

\[
(\hat{C}_{M,N}, \hat{C}_{N,M}) = \arg \min_{C_{M,N}} E_{M,N}(C_{M,N}) + E_{N,M}(C_{N,M}) \\
+ \beta \|C_{M,N} C_{N,M} - Id\|^2
\]

\textsuperscript{1} Coupled Functional Maps, D. Eynard et al., 3DV 2016.
Our approach:

\[
(\hat{C}_{M,N}, \hat{C}_{N,M}) = \arg \min E_{M,N}(C_{M,N}) + E_{N,M}(C_{N,M}) \\
+ \gamma_1\|X^A_{M,N} - C_{M,N}\|^2 + \gamma_2\|X^C_{M,N} - C_{M,N}\|^2
\]
Bi-directional Shape Matching

Our approach:

\[(\hat{C}_{M,N}, \hat{C}_{N,M}) = \arg\min E_{M,N}(C_{M,N}) + E_{N,M}(C_{N,M})
+ \gamma_1 \|C_{N,M}^T - C_{M,N}\|^2 + \gamma_2 \|\Lambda_N C_{M,N} - C_{N,M}^T \Lambda_M\|^2\]
Bi-directional Shape Matching
Bi-directional Shape Matching

Invertibility

Source Shape

Coverage = 5.0%

Coverage = 38.3%

Coverage = 49.8%

Regular Fmap
Coverage = 44.6%

[ERGB]
Coverage = 71.6%

Adjoint Regularization
Coverage = 81.5%

Regular Fmap + ICP

[ERGB] + ICP

Adjoint Regularization + ICP
Bi-directional Shape Matching

![Graphs showing the fraction of correspondences vs. geodesic error for different datasets and methods.](image)

The graphs illustrate the performance of various methods in matching shapes, with metrics such as the fraction of correspondences and geodesic error. Each dataset (FAUST, SCAPE, TOSCA) has different curves representing the performance of methods like Adjoint Regularization, [ERGB] + ICP, and Regular Fmaps + ICP.
Joint Shape Analysis

Where are the *jointly* most distorted areas in the collection?
Joint Shape Analysis

Previous approach\(^1\) (based on shape difference):

- Needs to choose a base shape – biased result;
- Not consistent.

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1 Analysis and Visualization of Maps Between Shapes, M. Ovsjanikov et al., CGF 2013.
Joint Shape Analysis

Our approach:

• The adjoint functional map reflects shape deformation and transform information across shapes at the same time.

• Extract a collection of consistent basis, using the approach of Wang et al.\textsuperscript{1};

• Find the jointly most distorted areas, by using the adjoint maps, rather than the functional maps within the same framework.

\textsuperscript{1} Image Co-segmentation via Consistent Functional Maps, F. Wang et al., ICCV 2013.
Joint Shape Analysis

Our approach:

- Find consistent $f = [f_1, f_2, ..., f_n]$ s.t. $\sum_{i,j} \| C_{ij} f_i - f_j \|_{L^2} = 0$
- In the space spanned by all the consistent $f$, maximize $\sum_{i,j} \| X_{ij} f_i - f_j \|_{L^2}$. 
Joint Shape Analysis

Our approach:

$X^A$
Joint Shape Analysis

Order the highlighted deformations with a unified measure.
Joint Shape Analysis
Something New Ongoing
Something New Ongoing
Conclusion

The adjoint map representation

• both transfers information and reflects deformation.
• can be obtained without extra effort.
Thanks for your attention!
Questions?

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Proposition. Let $X_{M,N}^A$ be the area-based adjoint operator of $C_{N,M}$, and let $\kappa = X_{N,M}^A(1_N)$, then we have

$$X_{M,N}^A(\kappa \cdot f) = C_{M,N}(f)$$