A bijective proof of the enumeration of maps in higher genus.

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LIX, École polytechnique

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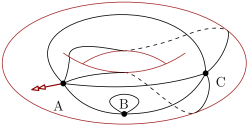
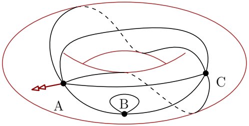
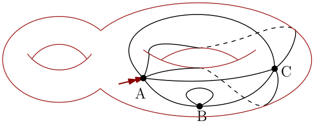
Maps



A rooted map

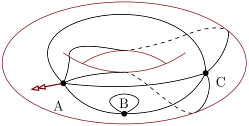


The same map



Not a map

Maps



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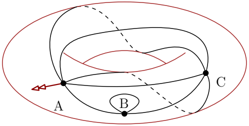
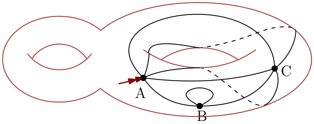


The same map



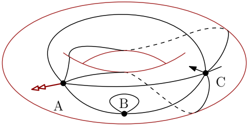
Not a map

Definition

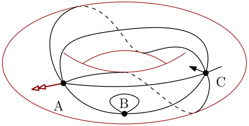
tree: *no cycle*

unicellular: *only one face*

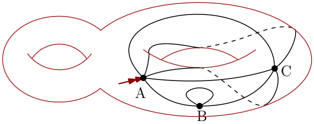
Maps



A blossoming map



The same map



Not a map

Definition

tree: *no cycle*

unicellular: *only one face*

blossoming: *with some (ingoing or outgoing) stems on corners*

Introduction

Theorem (Tutte 60's for $g = 0$, Bender Canfield 91 for $g > 0$)

For any $g \geq 0$, the generating series $M_g(z)$ of maps of genus g enumerated by edges is a rational function of z and $\sqrt{1 - 12z}$.

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Blossoming maps:

- in the plane: Schaeffer 97, Bouttier Di Francesco Guitter 02, Poulhalon Schaeffer 06, Bernardi 07...
- in higher genus: Bernardi Chapuy 11, Despres Gonçalves Leveque 17...

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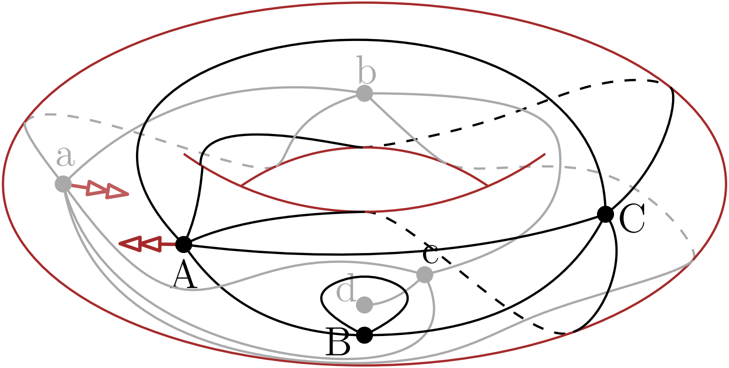
Labeled maps (mobiles):

- in the plane: Cori Vauquelin 81, Schaeffer 98, Bouttier Di Francesco Guitter 04...
- in higher genus: Chapuy Marcus Schaeffer 09...
- on non-orientable surfaces: Bettinelli 16, Chapuy Dolega 17...

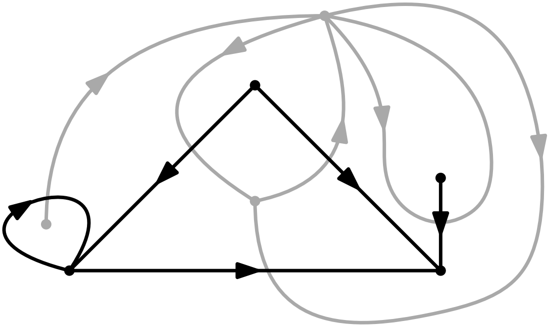
Sommaire

Classical constructions on maps



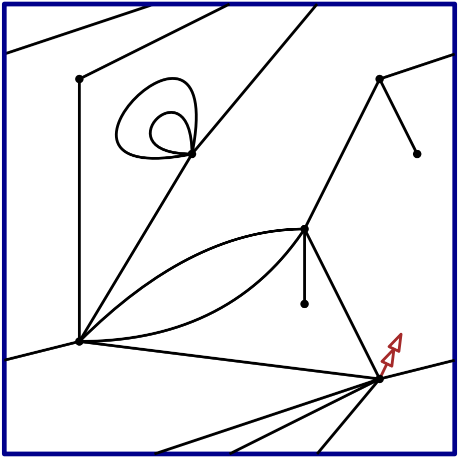
A map (black) and its dual (grey)

Classical constructions on maps



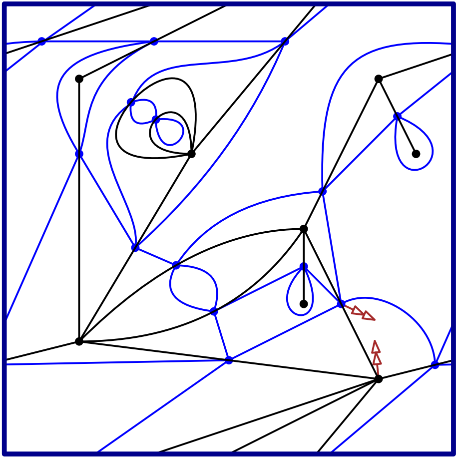
The convention for orienting a map (black) and its dual (grey)

Classical constructions on maps



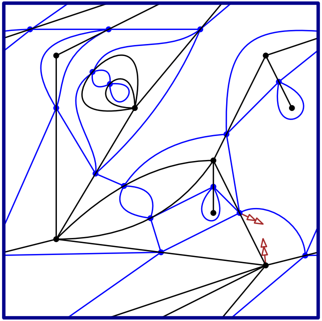
A classical representation of a toroidal map

Classical constructions on maps



The radial construction

Classical constructions on maps



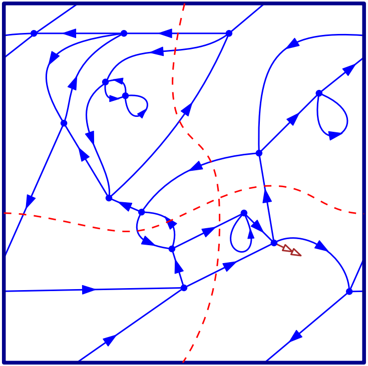
The radial construction

Proposition

There is a bijection between:

- *general maps of genus g with n edges, and*
- *4-valent bicolorable maps of genus g with n vertices.*

Bicolorable orientations

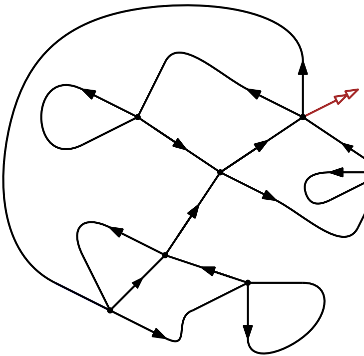


A 4-valent bicolorable toroidal map along with its dual-geodesic orientation

Definition

Bicolorable orientation: any dual cycle has as many edges going to the left and to the right.

Bicolorable orientations

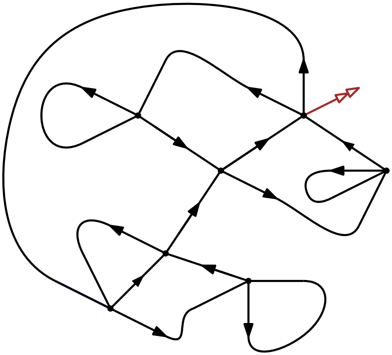


Definition

Eulerian map: all vertices have even degree

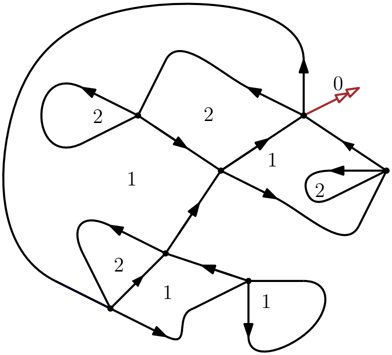
Eulerian orientation: all vertices have equal out- and in-degrees

Bicolorable orientations



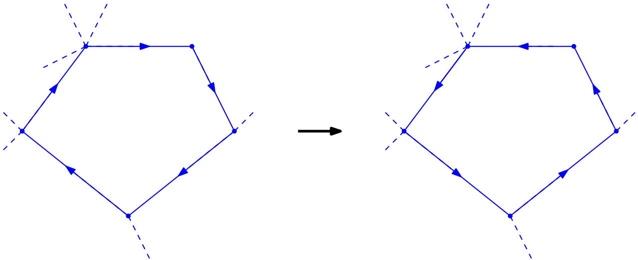
The orientation has no clockwise cycle...

Bicolorable orientations



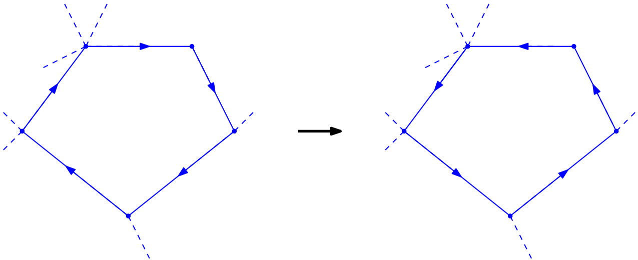
... It is the dual-geodesic orientation

Bicolorable orientations



A face-flip

Bicolorable orientations



A face-flip

Theorem (Propp 93)

The set of bicolorable orientations of a fixed map with face-flip as a cover relation forms a distributive lattice.
Its minimum is the dual-geodesic orientation.

Bicolorable orientations

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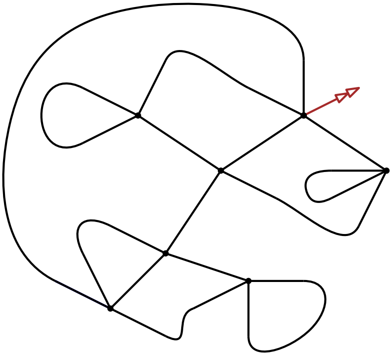
The set of bicolorable orientations of a fixed map with face-flip as a cover relation forms a distributive lattice.
Its minimum is the dual-geodesic orientation.

Corollary

The dual-geodesic orientation is the unique bicolorable orientation with no clockwise face.

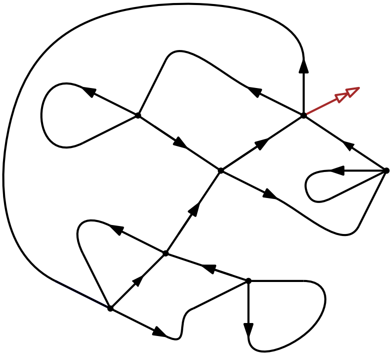
Sommaire

Opening a 4-valent planar map



A 4-valent (Eulerian) planar map

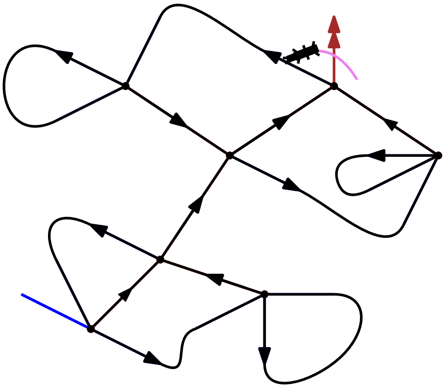
Opening a 4-valent planar map



With its dual-geodesic orientation

Opening a 4-valent planar map

- An unvisited ingoing edge is followed
- An unvisited outgoing edge is cut
- A visited leaf is ignored
- A visited outoing edge is followed



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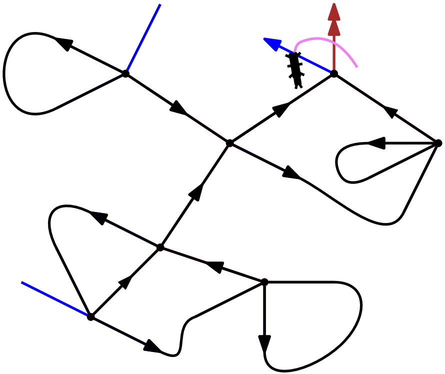

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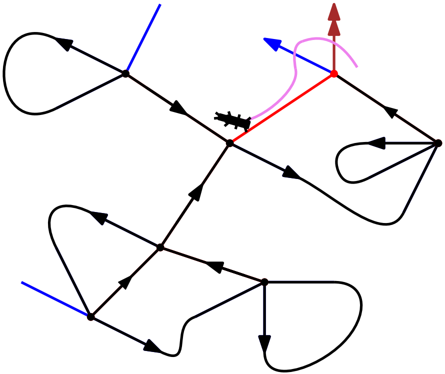

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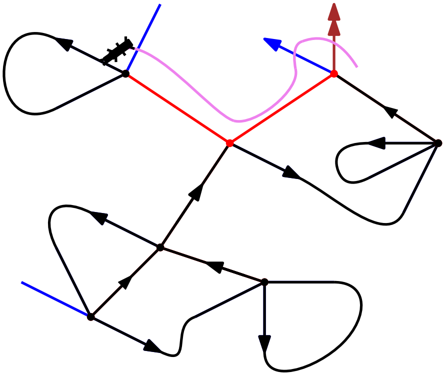

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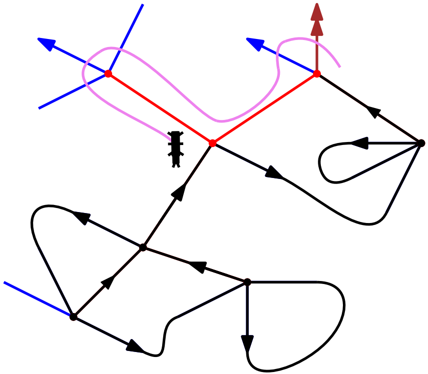

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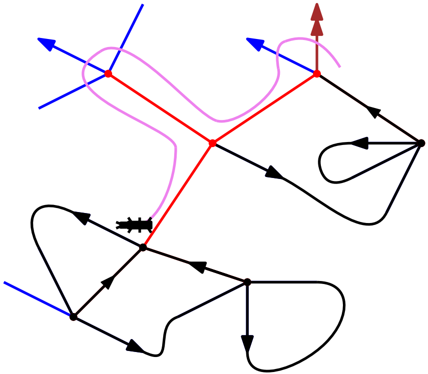

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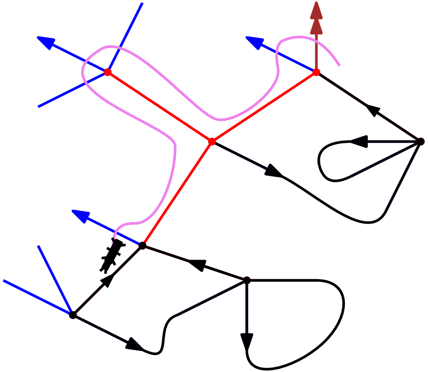

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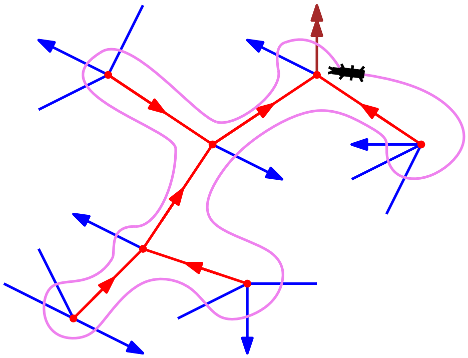

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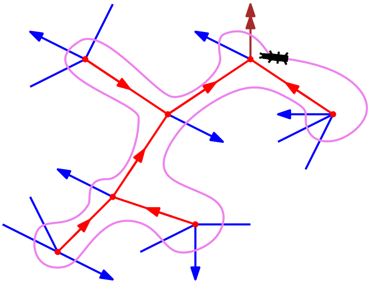
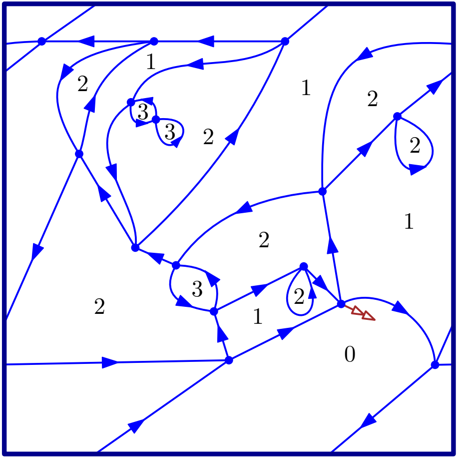


Theorem (Schaeffer 97)

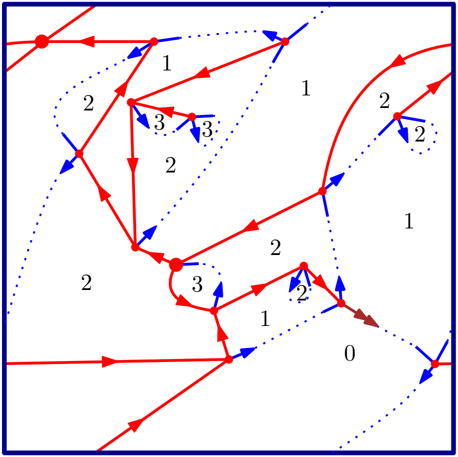
The opening algorithm is a bijection between 4-valent maps and well-rooted Eulerian 4-valent blossoming trees.

Opening a 4-valent bicolorable map



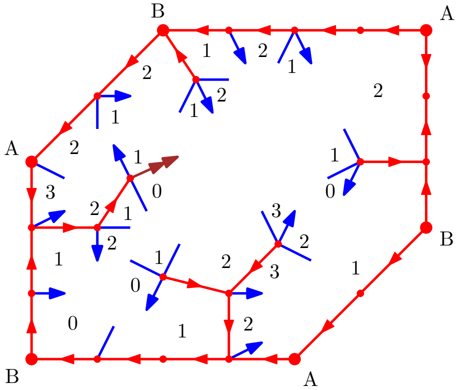
A 4-regular bicolorable map with dual-geodesic orientation

Opening a 4-valent bicolorable map



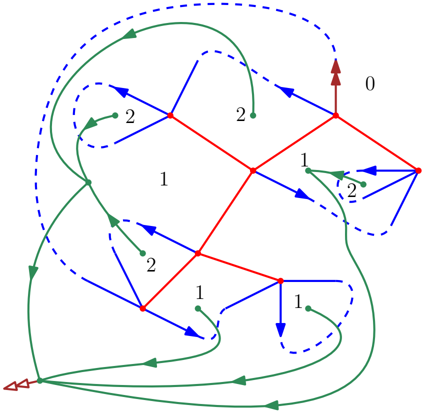
We apply the opening algorithm...

Opening a 4-valent bicolorable map



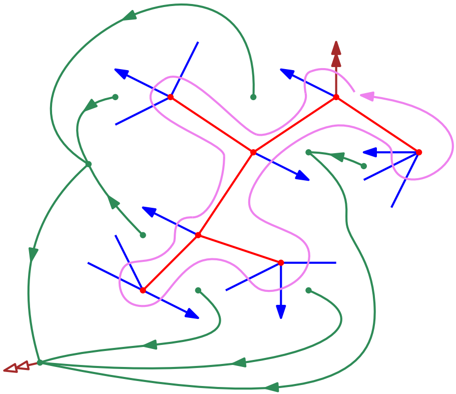
... And obtain a unicellular map

Opening a 4-valent bicolorable map



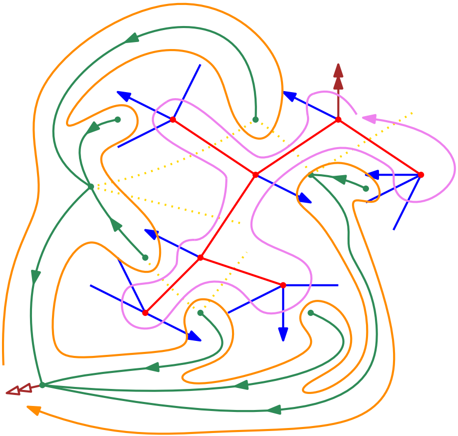
The obtained map is the dual of the leftmost breadth-first-search exploration tree

Opening a 4-valent bicolorable map



Indeed, the opening algorithm can be expressed as a walk on corners of the map, and hence has the same execution on a map and its dual

Opening a 4-valent bicolorable map



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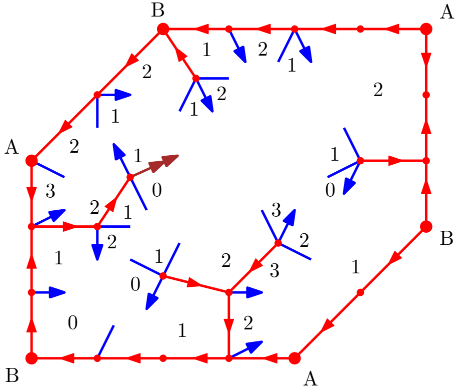
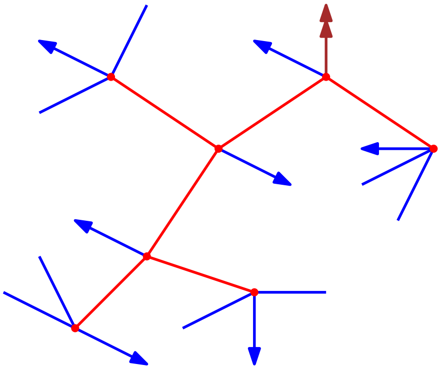


Theorem (L.)

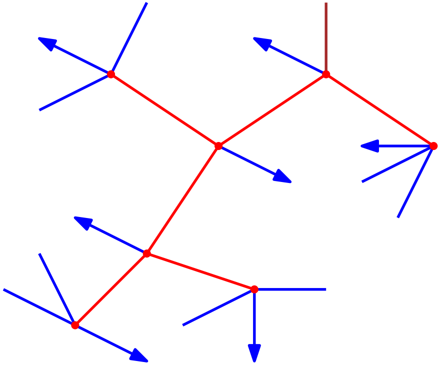
The opening algorithm is a bijection between bicolorable 4*-valent map and well-rooted well-oriented well-labeled* 4*-valent unicellular maps.*

Closing an Eulerian 4-valent blossoming tree



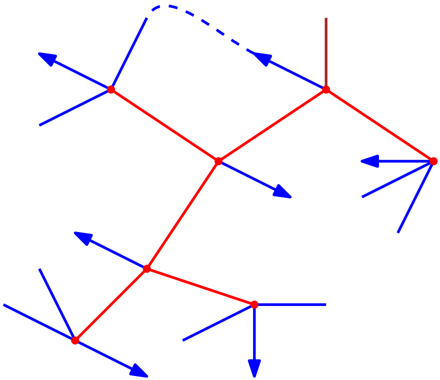
A 4-valent Eulerian rooted tree

Closing an Eulerian 4-valent blossoming tree



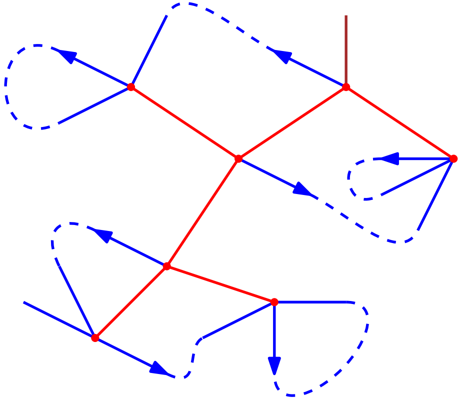
The root is reversed

Closing an Eulerian 4-valent blossoming tree



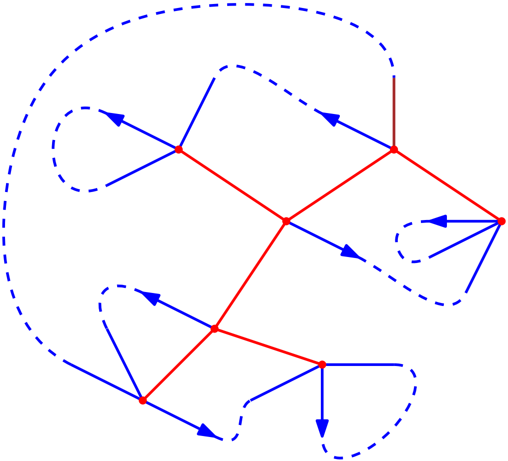
A bud and a leaf following one another are merged

Closing an Eulerian 4-valent blossoming tree



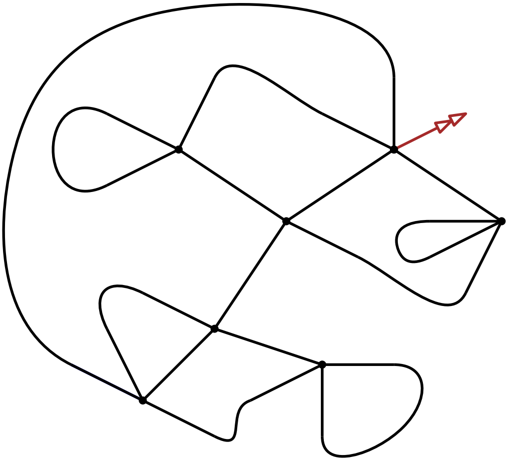
This is repeated until no such pair exists

Closing an Eulerian 4-valent blossoming tree



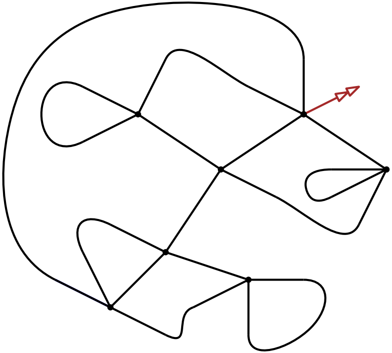
In the end the 2 remaining leaves are merged

Closing an Eulerian 4-valent blossoming tree



We obtain a map

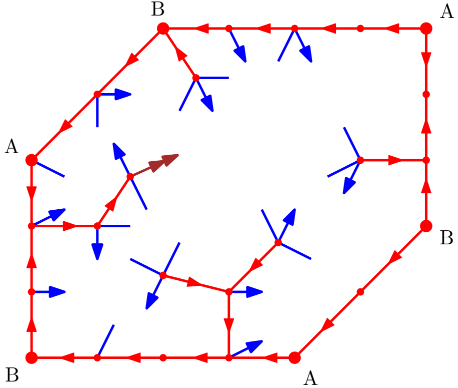
Closing an Eulerian 4-valent blossoming tree



Theorem (Schaeffer 97)

The closing algorithm is the inverse bijection of the opening algorithm.

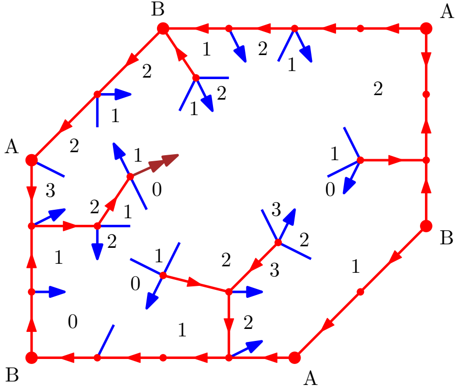
Closing a well-rooted well-labeled well-oriented 4-valent unicellular blossoming map



Definition

well-oriented: in a tour of the face, each edge is first followed backward.

Closing a well-rooted well-labeled well-oriented 4-valent unicellular blossoming map

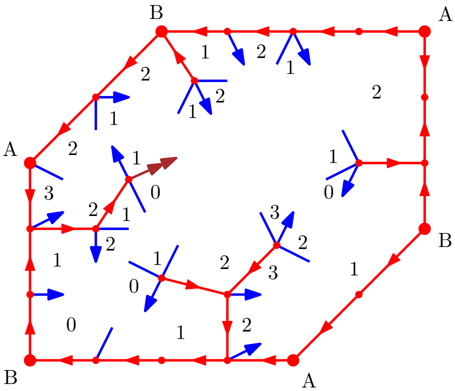


Definition

well-oriented: in a tour of the face, each edge is first followed backward.

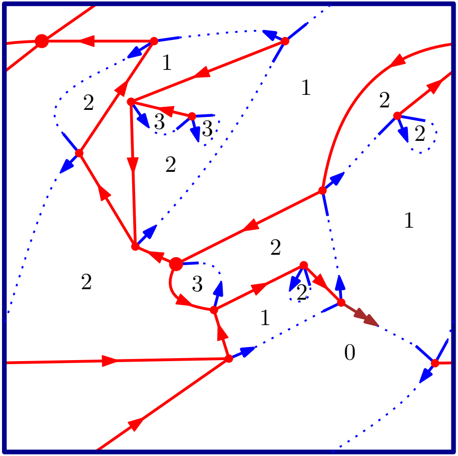
well-labeled: the difference between adjacent labels correspond to the crossed edges or stems.

Closing a well-rooted well-labeled well-oriented 4-valent unicellular blossoming map



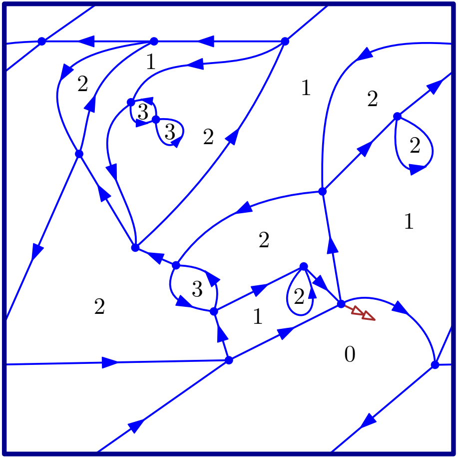
A special map (see the never-ending title)

Closing a well-rooted well-labeled well-oriented 4-valent unicellular blossoming map



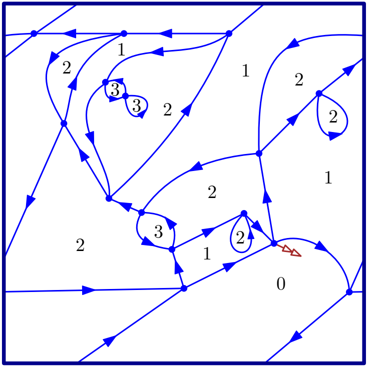
Matching stems

Closing a well-rooted well-labeled well-oriented 4-valent unicellular blossoming map



We again obtain a map, with its dual-geodesic orientation

Closing a well-rooted well-labeled well-oriented 4-valent unicellular blossoming map

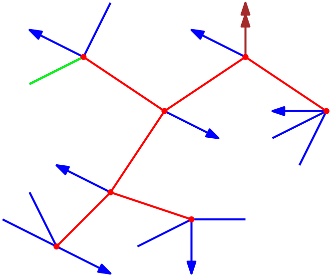


Theorem (L.)

The closing algorithm is the inverse bijection of the opening algorithm.

Sommaire

Well-rooted is an inconveniently global condition

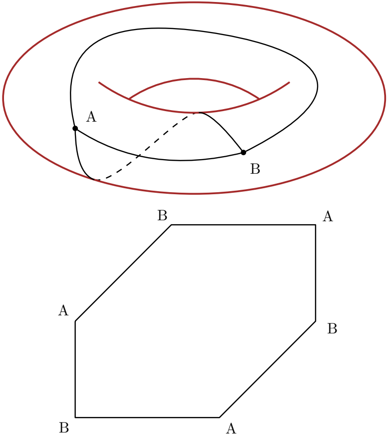


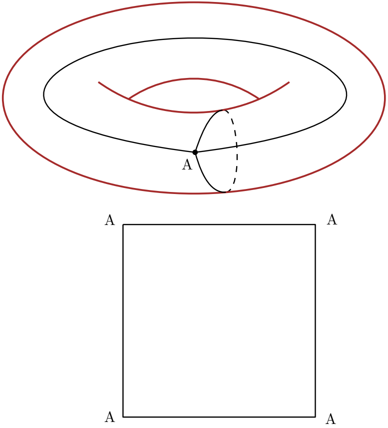
Theorem (Schaeffer 97 for $g = 0$, L. for $g > 0$)

For a fixed interior map m with n leaves, there is a 2-to-$(n+1)$ map from

- *well-rooted well-labeled well-oriented 4-regular unicellular map with interior map m, to*
- *rooted well-labeled well-oriented 4-regular unicellular map with interior map m (which has n leaves).*

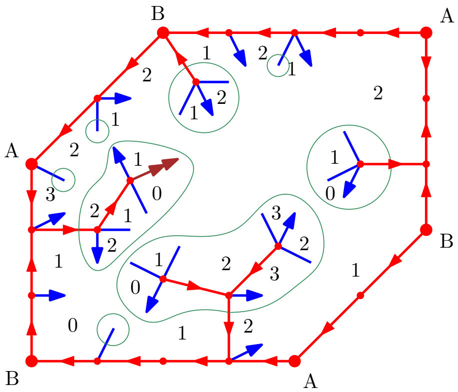
The structure of unicellular maps





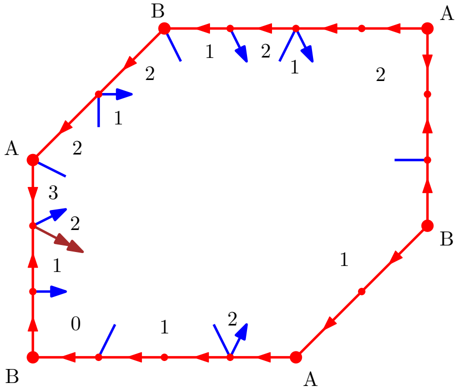
The schemes of genus 1

Pruning the map



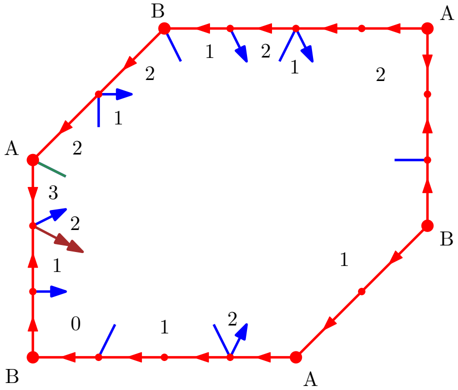
The opened map contains treelike parts

Pruning the map



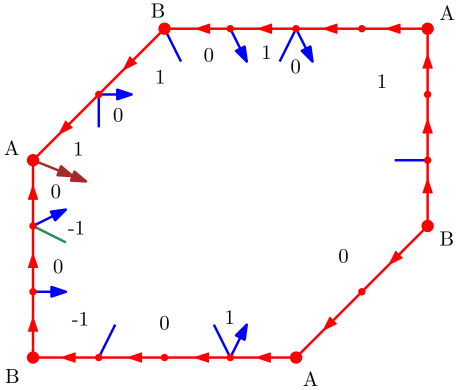
These treelike parts are removed

Rerooting on the scheme



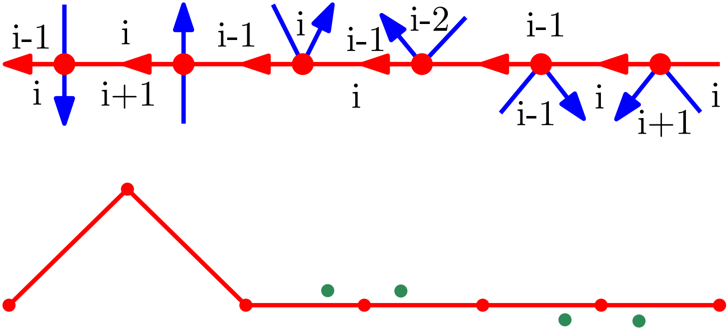
The pruned map...

Rerooting on the scheme



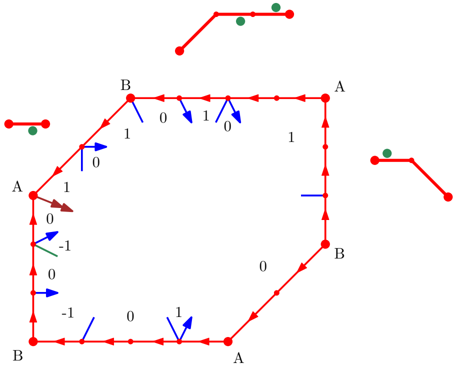
...is rerooted on the scheme

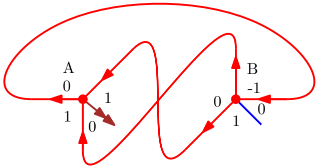
Replace branches by decorated Motzkin paths



There are 6 type of vertices of interior degree 2, each of which can be represented by a decorated Motzkin step

Replace branches by decorated Motzkin paths





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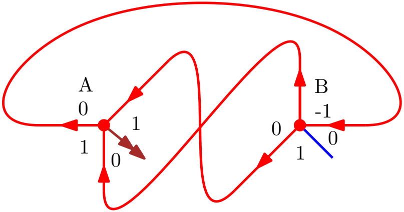

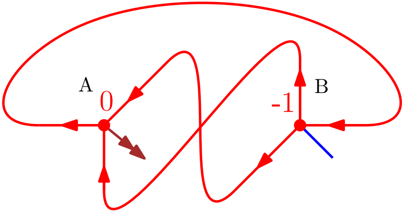

What about generating series?

Notation

The series of decorated Motzkin bridges is $B(z)$.
The series of decorated Motzkin positive bridges followed by a downstep is $D(z)$.

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Lemma (Chapuy Marcus Schaeffer 09)

A series is rational in z if and only if it is rational and symmetric in D.

A series F in D is symmetric in D if $\overline{F(D)} = F(D)$
(avec $\overline{F(D)} := F(D^{-1})$).

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Lemma (Chapuy Marcus Schaeffer 09)

A series is rational in z if and only if it is rational and symmetric in D.

The generating series of Motzkin paths from height i to j is: $B \cdot D^{|i-j|}$.

So, what do we have?

We derive from the previous work that the generating series $M^s(t)$ of all maps (counted by number of edges) with a given (unlabeled) scheme s is:

$$M^s(t) = t^{2g-1} \cdot \frac{2}{t} \int_0^t z \cdot \frac{\partial T}{\partial z}(z) \cdot \frac{\partial R^s}{\partial z}(T(z))dz,$$

where T is the series of trees, and R^s the series of scheme-rooted well-oriented well-labeled pruned unicellular maps with unlabeled scheme s.

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Theorem (L.)

The generating series $M^s(t)$ is a rational function of $T(t)$.

So, what do we have?

Theorem (L.)

The generating series $M^s(t)$ is a rational function of $T(t)$.

Since the set of scheme $\mathcal{S}_g$ is finite for any fixed g, it implies that $M_g(t) = \sum_{s \in \mathcal{S}_g} M^s(t)$ is rational in $T(t)$, and proves our main theorem.

Theorem (Tutte 60's for $g = 0$, Bender Canfield 91 for $g > 0$)

For any g, the generating series $M_g(t)$ is a rational function of $T(t)$.

What's left to do

Lemma (L.)

$R^s(z)$ is rational and symmetric in D.

Obtaining an equation for a given scheme

$$R^l = \prod_{(v_i,v_j)\in E_l} B \cdot D^{|h_i - h_j|}$$

- l is a labeled scheme
- E_l is the set of its edges
- h_i is the label of vertex v_i.

Obtaining an equation for a given scheme

$$
\begin{aligned}
R^l &= \prod_{(v_i,v_j)\in E_l} B \cdot D^{|h_i-h_j|} \\
R^s &= \sum_{\substack{h_1\cdots h_{n_v}\in\mathbb{N} \\ \min(h_1,\cdots,h_{n_v})=0}} \prod_{(v_i,v_j)\in E_s} B \cdot D^{|h_i-h_j|}
\end{aligned}
$$

- s is an unlabeled scheme.
- Labels are defined up to translation, so we decide to force the minimal label to be 0.

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&= B^{|E_l|} \cdot \sum_{S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_k} \prod_{i=1}^{k-1} \frac{D^{Cut(S_i)}}{1-U^{Cut(S_i)}}
\end{aligned}
$$

- $\emptyset = S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_k = V_s$ is the ordered partition corresponding to the ordering of labels.
- $Cut(S)$ is the number of edges going from S to $\overline{S}$.

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&= B^{|E_l|} \cdot \sum_{S_0\subsetneq S_1\subsetneq\cdots\subsetneq S_k} \prod_{i=1}^{k-1} \Phi(S_i)
\end{aligned}
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- $\Phi(S) = \frac{D^{Cut(S)}}{1-D^{Cut(S)}}$.

Obtaining an equation for a given scheme

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R^l &= \prod_{(v_i,v_j)\in E_l} B \cdot D^{|h_i - h_j|} \\
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&= B^{|E_l|} \cdot \sum_{S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_k} \prod_{i=1}^{k-1} \Phi(S_i) \\
&= B^{|E_l|} \cdot \sum_{\pi} \prod_{i=1}^{k-1} \Phi(S_i)
\end{aligned}
$$

- The ordered partition is called π instead of $S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_k$.

Proving that this expression is symmetric

$$R^s = B^{|E_I|} \quad \cdot \sum_{\pi} \prod_{i=1}^{k-1} \Phi(S_i)$$

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$$\overline{R^s} = (-1)^{|E_I|} \cdot B^{|E_I|} \quad \cdot \sum_{\pi} \left((-1)^{k-1} \cdot \prod_{i=1}^{k-1} (1 + \Phi(S_i)) \right)$$

- B is antisymmetric.
- $\overline{\Phi(S)} = -(1 + \Phi(S))$.

Proving that this expression is symmetric

$$\begin{aligned}
R^s &= B^{|E_I|} &&\cdot \sum_{\pi} \prod_{i=1}^{k-1} \Phi(S_i) \\
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\end{aligned}$$

- $\mu < \pi$ means that π refines μ as an ordered partition.
- We apply the inclusion-exclusion principle.

Proving that this expression is symmetric

$$
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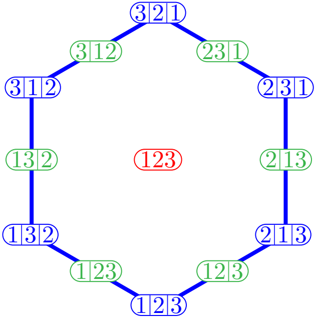
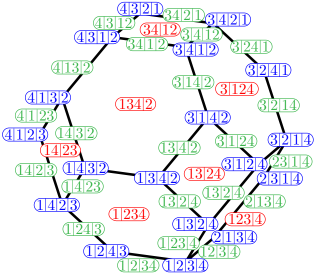
&= (-1)^{|E_I|} \cdot B^{|E_I|} &&\cdot \sum_{\pi} \prod_{i=1}^{k(\pi)-1} \Phi(S_i(\pi)) \cdot \sum_{\pi < \rho} \left((-1)^{k(\rho)-1} \right)
\end{aligned}
$$

- We swap the two sumations.

Proving that this expression is symmetric





- ordered partitions are faces of the permutahedron.
- Euler-Poincaré's formula states:

$$0 = \sum_{i=-1}^{d} (-1)^i f_i(P),$$

where P is a polytope of degree d with $f_i(P)$ faces of degree i.

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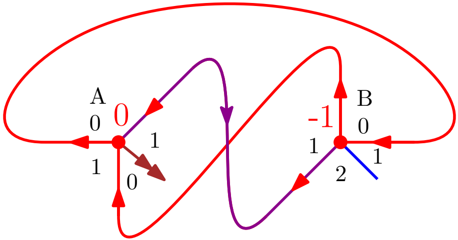
The end

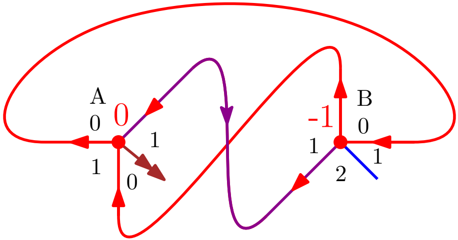
Thanks for your attention!

The offset graph



An offset edge (purple)

The offset graph



Theorem (L.)

The offset graph of a scheme-rooted well-oriented well-labeled unicellular map is acyclic.

The offset graph

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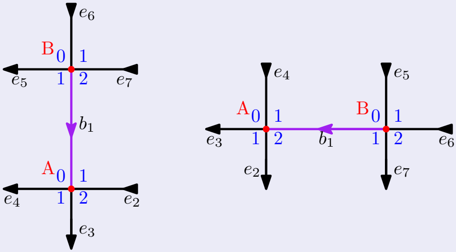
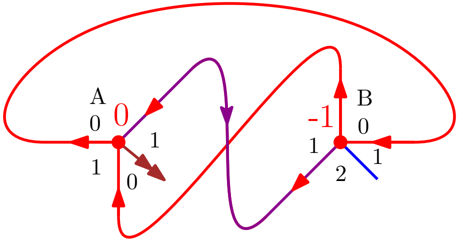
Proof.



The two possible case for the first step of an offset cycle

The offset graph



With an offset graph:

$$R^s = B^{|E_l|} \cdot \sum_{S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_k} \left(\prod_{i=1}^{k-1} \Phi(S_i) \right) \cdot D^{n_t + n_a - n_i}.$$

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By consequence:

$$\overline{R^s} = (-1)^{|E_l|} \cdot B^{|E_l|} \cdot \sum_{\pi} \prod_{i=1}^{k(\pi)-1} \Phi(S_i(\pi)) \cdot \sum_{\pi < \rho} \left((-1)^{k(\rho)-1} \cdot D^{-n_t(\rho) - n_a(\rho) + n_i(\rho)} \right)$$

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We need to prove the following lemma:

Lemma (L.)

$$\sum_{\pi^{-1} < \rho} \left((-1)^{k(\rho)-1} \cdot U^{-n_t(\rho) - n_a(\rho) + n_i(\rho)} \right) = (-1)^{|E_s|} \cdot D^{n_t(\pi) + n_a(\pi) - n_i(\pi)}$$