A bijective proof of the enumeration of maps in higher genus.

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A rooted map

The same map

Not a map
Maps

A rooted map

The same map

Not a map

Definition

- **tree**: no cycle
- **unicellular**: only one face
Maps

A blossoming map  The same map  Not a map

Definition

tree: no cycle
unicellular: only one face
blossoming: with some (ingoing or outgoing) stems on corners
Theorem (Tutte 60’s for $g = 0$, Bender Canfield 91 for $g > 0$)

For any $g \geq 0$, the generating series $M_g(z)$ of maps of genus $g$ enumerated by edges is a rational function of $z$ and $\sqrt{1 - 12z}$.
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Blossoming maps:
- in the plane: Schaeffer 97, Bouttier Di Francesco Guitter 02, Poulhalon Schaeffer 06, Bernardi 07...
- in higher genus: Bernardi Chapuy 11, Despres Gonçalves Leveque 17...
Introduction

Theorem (Tutte 60’s for $g = 0$, Bender Canfield 91 for $g > 0$)

For any $g \geq 0$, the generating series $M_g(z)$ of maps of genus $g$ enumerated by edges is a rational function of $z$ and $\sqrt{1 - 12z}$.

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Labeled maps (mobiles):
- in the plane: Cori Vauquelin 81, Schaeffer 98, Bouttier Di Francesco Guitter 04...
- in higher genus: Chapuy Marcus Schaeffer 09...
- on non-orientable surfaces: Bettinelli 16, Chapuy Dolega 17...
Sommaire

1 Orientations of a map
   - Definitions
   - Structure

2 Opening and closing maps
   - The opening of a map
   - The closure of a blossoming map

3 Enumeration and rationality
   - Reducing a map to a scheme
   - Analysing a scheme
   - Rationality
A map (black) and its dual (grey)
The convention for orienting a map (black) and its dual (grey)
A classical representation of a toroidal map
Classical constructions on maps

The radial construction
Classical constructions on maps

The radial construction

Proposition

There is a bijection between:

- general maps of genus $g$ with $n$ edges, and
- 4-valent bicolorable maps of genus $g$ with $n$ vertices.
Bicolorable orientations

A 4-valent bicolorable toroidal map along with its dual-geodesic orientation

Definition

Bicolorable orientation: any dual cycle has as many edges going to the left and to the right.
Bicolorable orientations

**Definition**

**Eulerian map**: all vertices have even degree

**Eulerian orientation**: all vertices have equal out- and in-degrees
Bicolorable orientations

The orientation has no clockwise cycle...
Bicolorable orientations

... It is the dual-geodesic orientation
Bicolorable orientations

A face-flip

The set of bicolorable orientations of a fixed map with face-flip as a cover relation forms a distributive lattice. Its minimum is the dual-geodesic orientation.
Bicolorable orientations

A face-flip

**Theorem (Propp 93)**

The set of bicolorable orientations of a fixed map with face-flip as a cover relation forms a distributive lattice. Its minimum is the dual-geodesic orientation.
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The set of bicolorable orientations of a fixed map with face-flip as a cover relation forms a distributive lattice.
Its minimum is the dual-geodesic orientation.

Corollary

The dual-geodesic orientation is the unique bicolorable orientation with no clockwise face.
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3. Enumeration and rationality
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   - Analysing a scheme
   - Rationality
Opening a 4-valent planar map

A 4-valent (Eulerian) planar map
Opening a 4-valent planar map

With its dual-geodesic orientation
Opening a 4-valent planar map

- An unvisited ingoing edge is followed
- An unvisited outgoing edge is cut
- A visited leaf is ignored
- A visited outgoing edge is followed
Opening a 4-valent planar map

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Opening a 4-valent planar map

Theorem (Schaeffer 97)

The opening algorithm is a bijection between 4-valent maps and well-rooted Eulerian 4-valent blossoming trees.
Opening a 4-valent bicolorable map

A 4-regular bicolorable map with dual-geodesic orientation
Opening a 4-valent bicolorable map

We apply the opening algorithm...
Opening a 4-valent bicolorable map

... And obtain a unicellular map
Opening a 4-valent bicolorable map

The obtained map is the dual of the leftmost breadth-first-search exploration tree
Opening a 4-valent bicolorable map

Indeed, the opening algorithm can be expressed as a walk on corners of the map, and hence has the same execution on a map and its dual.
Opening a 4-valent bicolorable map

Indeed, the opening algorithm can be expressed as a walk on corners of the map, and hence has the same execution on a map and its dual.
Theorem (L.)

The opening algorithm is a bijection between bicolorable 4-valent map and well-rooted well-oriented well-labeled 4-valent unicellular maps.
Closing an Eulerian 4-valent blossoming tree

A 4-valent Eulerian rooted tree
Closing an Eulerian 4-valent blossoming tree

The root is reversed
A bud and a leaf following one another are merged
Closing an Eulerian 4-valent blossoming tree

This is repeated until no such pair exists
Closing an Eulerian 4-valent blossoming tree

In the end the 2 remaining leaves are merged
We obtain a map
Theorem (Schaeffer 97)

*The closing algorithm is the inverse bijection of the opening algorithm.*
Closing a well-rooted well-labeled well-oriented 4-valent unicellular blossoming map

Definition

well-oriented: in a tour of the face, each edge is first followed backward.
Closing a well-rooted well-labeled well-oriented 4-valent unicellular blossoming map

**Definition**

**well-oriented:** in a tour of the face, each edge is first followed backward.

**well-labeled:** the difference between adjacent labels correspond to the crossed edges or stems.
Closing a well-rooted well-labeled well-oriented 4-valent unicellular blossoming map

A special map (see the never-ending title)
Closing a well-rooted well-labeled well-oriented 4-valent unicellular blossoming map

Matching stems
Closing a well-rooted well-labeled well-oriented 4-valent unicellular blossoming map

We again obtain a map, with its dual-geodesic orientation
Closing a well-rooted well-labeled well-oriented 4-valent unicellular blossoming map

Theorem (L.)

*The closing algorithm is the inverse bijection of the opening algorithm.*
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Well-rooted is an inconveniently global condition

Theorem (Schaeffer 97 for $g = 0$, L. for $g > 0$)

For a fixed interior map $m$ with $n$ leaves, there is a 2-to-$(n + 1)$ map from

- well-rooted well-labeled well-oriented 4-regular unicellular map with interior map $m$, to

- rooted well-labeled well-oriented 4-regular unicellular map with interior map $m$ (which has $n$ leaves).
The structure of unicellular maps

The schemes of genus 1
Pruning the map

The opened map contains treelike parts
These treelike parts are removed
Rerooting on the scheme

The pruned map...
Rerooting on the scheme

...is rerooted on the scheme
Replace branches by decorated Motzkin paths

There are 6 type of vertices of interior degree 2, each of which can be represented by a decorated Motzkin step
Replace branches by decorated Motzkin paths
Replace branches by decorated Motzkin paths
What about generating series?

**Notation**

-The series of decorated Motzkin bridges is \( B(z) \).
-The series of decorated Motzkin positive bridges followed by a downstep is \( D(z) \).
What about generating series?

**Notation**

The series of decorated Motzkin bridges is $B(z)$.
The series of decorated Motzkin positive bridges followed by a downstep is $D(z)$.

They satisfy $D = z(1 + 4D + D^2)$ and $B = 1 + 4zB + 2zDB$. 
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Hence $B = \frac{1+4D+D^2}{1-D^2}$ and $z = \frac{1}{D-1+4+D}$. 
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Hence $B = \frac{1+4D+D^2}{1-D^2}$ and $z = \frac{1}{D^{-1}+4+D}$.

**Lemma (Chapuy Marcus Schaeffer 09)**

A series is rational in $z$ if and only if it is rational and symmetric in $D$.

A series $F$ in $D$ is symmetric in $D$ if $\overline{F(D)} = F(D)$ (avec $\overline{F(D)} := F(D^{-1})$).
What about generating series?

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**Lemma (Chapuy Marcus Schaeffer 09)**

A series is rational in $z$ if and only if it is rational and symmetric in $D$.

The generating series of Motzkin paths from height $i$ to $j$ is: $B \cdot D^{|i-j|}$. 
So, what do we have?

We derive from the previous work that the generating series $M^s(t)$ of all maps (counted by number of edges) with a given (unlabeled) scheme $s$ is:

$$M^s(t) = t^{2g-1} \cdot \frac{2}{t} \int_0^t z \cdot \frac{\partial T}{\partial z}(z) \cdot \frac{\partial R^s}{\partial z}(T(z)) \, dz,$$

where $T$ is the series of trees, and $R^s$ the series of scheme-rooted well-oriented well-labeled pruned unicellular maps with unlabeled scheme $s$. 

Theorem (L.)

The generating series $M^s(t)$ is a rational function of $T(t)$. 

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**Theorem (L.)**

The generating series $M^s(t)$ is a rational function of $T(t)$.
Theorem (L.)

The generating series \( M^s(t) \) is a rational function of \( T(t) \).

Since the set of scheme \( S_g \) is finite for any fixed \( g \), it implies that
\[
M_g(t) = \sum_{s \in S_g} M^s(t)
\]
is rational in \( T(t) \), and proves our main theorem.

Theorem (Tutte 60’s for \( g = 0 \), Bender Canfield 91 for \( g > 0 \))

For any \( g \), the generating series \( M_g(t) \) is a rational function of \( T(t) \).
Lemma (L.)

\[ R^s(z) \text{ is rational and symmetric in } D. \]
Obtaining an equation for a given scheme

\[ R^l = \prod_{(v_i, v_j) \in E_l} B \cdot D|h_i - h_j| \]

- \( l \) is a labeled scheme
- \( E_l \) is the set of its edges
- \( h_i \) is the label of vertex \( v_i \).
Obtaining an equation for a given scheme

\[ R^l = \prod_{(v_i, v_j) \in E_l} B \cdot D |h_i - h_j| \]

\[ R^s = \sum_{h_1 \cdots h_{n_v} \in \mathbb{N}} \prod_{min(h_1, \cdots, h_{n_v}) = 0} B \cdot D |h_i - h_j| \]

- \( s \) is an unlabeled scheme.
- Labels are defined up to translation, so we decide to force the minimal label to be 0.
Obtaining an equation for a given scheme

\[ R^l = \prod_{(v_i, v_j) \in E_l} B \cdot D^{|h_i - h_j|} \]

\[ R^s = \sum_{h_1 \cdots h_{n_v} \in \mathbb{N}} \prod_{(v_i, v_j) \in E_s} B \cdot D^{|h_i - h_j|} \]

\[ = B |E_l| \cdot \sum_{S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_k} \prod_{i=1}^{k-1} \frac{D \text{Cut}(S_i)}{1 - U \text{Cut}(S_i)} \]

- \( \emptyset = S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_k = V_s \) is the ordered partition corresponding to the ordering of labels.

- \( \text{Cut}(S) \) is the number of edges going from \( S \) to \( \overline{S} \).
Obtaining an equation for a given scheme

\[ R^l = \prod_{(v_i, v_j) \in E_l} B \cdot D^{|h_i - h_j|} \]

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\[ = B |E_l| \cdot \sum_{S_0 \subset S_1 \subset \ldots \subset S_k} \prod_{i=1}^{k-1} \Phi(S_i) \]

\[ \Phi(S) = \frac{D \text{Cut}(S)}{1 - D \text{Cut}(S)}. \]
Obtaining an equation for a given scheme

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\[ = B^{|E_l|} \cdot \sum_{\pi} \prod_{i=1}^{k-1} \Phi(S_i) \]

- The ordered partition is called \( \pi \) instead of \( S_0 \subset S_1 \subset \cdots \subset S_k \).
Proving that this expression is symmetric

\[ R^s = B^{|E_i|} \cdot \sum_{\pi} \prod_{i=1}^{k-1} \Phi(S_i) \]
Proving that this expression is symmetric

\[ R^s = B^{|E_l|} \cdot \sum_{\pi} \prod_{i=1}^{k-1} \Phi(S_i) \]

\[ \overline{R^s} = (-1)^{|E_l|} \cdot B^{|E_l|} \cdot \sum_{\pi} \left( (-1)^{k-1} \cdot \prod_{i=1}^{k-1} (1 + \Phi(S_i)) \right) \]

- \( B \) is antisymmetric.
- \( \Phi(S) = -(1 + \Phi(S)) \).
Proving that this expression is symmetric

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\[ = (-1)^{|E_i|} \cdot B^{|E_i|} \cdot \sum_{\pi} \left( (-1)^{k(\pi)-1} \cdot \sum_{\mu<\pi} \prod_{i=1}^{k(\mu)-1} \Phi(S_i(\mu)) \right) \]

- \( \mu < \pi \) means that \( \pi \) refines \( \mu \) as an ordered partition.
- We apply the inclusion-exclusion principle.
Proving that this expression is symmetric

\[ R^s = B|E_i| \cdot \sum_{\pi} \prod_{i=1}^{k-1} \Phi(S_i) \]

\[ \overline{R^s} = (-1)^{|E_i|} \cdot B|E_i| \cdot \sum_{\pi} \left( (-1)^{k-1} \cdot \prod_{i=1}^{k-1} \left( 1 + \Phi(S_i) \right) \right) \]

\[ = (-1)^{|E_i|} \cdot B|E_i| \cdot \sum_{\pi} \left( (-1)^{k(\pi)-1} \cdot \sum_{\mu<\pi} \prod_{i=1}^{k(\mu)-1} \Phi(S_i(\mu)) \right) \]

\[ = (-1)^{|E_i|} \cdot B|E_i| \cdot \sum_{\pi} \prod_{i=1}^{k(\pi)-1} \Phi(S_i(\pi)) \cdot \sum_{\pi<\rho} \left( (-1)^{k(\rho)-1} \right) \]

- We swap the two summations.
Proving that this expression is symmetric

ordered partitions are faces of the permutahedron.

Euler-Poincaré’s formula states:

$$0 = \sum_{i=-1}^{d} (-1)^i f_i(P),$$

where $P$ is a polytope of degree $d$ with $f_i(P)$ faces of degree $i$. 
Proving that this expression is symmetric

\[ R^s = B^{|E_l|} \cdot \sum_{\pi} \prod_{i=1}^{k-1} \Phi(S_i) \]

\[ \overline{R^s} = (-1)^{|E_l|} \cdot B^{|E_l|} \cdot \sum_{\pi} \prod_{i=1}^{k(\pi)-1} \Phi(S_i(\pi)) \cdot \sum_{\pi<\rho} \left((-1)^{k(\rho)-1}\right) \]

- ordered partitions are faces of the permutahedron.
- Euler-Poincaré’s formula states:

\[ 0 = \sum_{i=-1}^{d} (-1)^i f_i(P), \]

where \( P \) is a polytope of degree \( d \) with \( f_i(P) \) faces of degree \( i \).
Thanks for your attention!
The offset graph

An offset edge (purple)
The offset graph

Theorem (L.)

The offset graph of a scheme-rooted well-oriented well-labeled unicellular map is acyclic.
Theorem (L.)

The offset graph of a scheme-rooted well-oriented well-labeled unicellular map is acyclic.

Proof.

The two possible case for the first step of an offset cycle
With an offset graph:

\[ R^s = B^{|E_l|} \cdot \sum_{S_0 \subsetneq S_1 \subsetneq \ldots \subsetneq S_k} \left( \prod_{i=1}^{k-1} \Phi(S_i) \right) \cdot D^{n_t+n_a-n_i}. \]
The offset graph

With an offset graph:

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By consequence:

\[ \overline{R^s} = \left( -1 \right)^{|E_l|} B^{|E_l|} \sum_k \prod_{\pi} \Phi(S_i(\pi)) \cdot \sum_{\pi < \rho} \left( (-1)^{k(\rho)-1} \cdot D^{-n_t(\rho)-n_a(\rho)+n_i(\rho)} \right). \]
The offset graph

With an offset graph:

$$R^s = B^{|E_i|} \cdot \sum_{S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_k} \left( \prod_{i=1}^{k-1} \Phi(S_i) \right) \cdot D^{n_t+n_a-n_i}.$$ 

By consequence:

$$\overline{R^s} = (-1)^{|E_i|} B^{|E_i|} \sum_{\pi} \prod_{i=1}^{k(\pi)-1} \Phi(S_i(\pi)) \cdot \sum_{\pi < \rho} \left( (-1)^{k(\rho)-1} \cdot D^{-n_t(\rho)-n_a(\rho)+n_i(\rho)} \right)$$

We need to prove the following lemma:

**Lemma (L.)**

$$\sum_{\pi^{-1} < \rho} \left( (-1)^{k(\rho)-1} \cdot U^{-n_t(\rho)-n_a(\rho)+n_i(\rho)} \right) = (-1)^{|E_s|} \cdot D^{n_t(\pi)+n_a(\pi)-n_i(\pi)}$$

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