Reach, metric distortion and variation of tangent space

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Motivation

- Reach,
- metric distortion,
- variation of tangent space

These are general geometric properties encountered in the proofs of several theorems that state topological faithful reconstructions of manifolds as well as more general subsets of Euclidean space by simplicial complexes.
Medial Axis and Reach

closed set C
Medial Axis and Reach

point in $C$ closest to $p$
Medial Axis and Reach

points in $C$ closest to $p$

closed set $C$
Medial Axis and Reach

Medial axis of a closed set $C$ is
« set of points who have at least two closest points in $C$ »
Medial Axis and Reach

**Medial axis** of a closed set $C$ is « set of points who have at least two closest points in $C$ »
Medial axis of a closed set $C$ is « set of points who have at least two closest points in $C$ »

Reach of a closed set $C$ « infimum of distances between $C$ and its medial axis »
The Reach

- Introduced by Herbert Federer (Curvature Measures 1959): classes of sets with positive reach allow to define curvature measures beyond smooth case.
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- Introduced by **Herbert Federer** (Curvature Measures 1959): classe of **sets with positive reach** allow to define curvature measures beyond smooth case.

- Used again in the context of **manifold reconstruction with topological guarantees** : Amenta et al. (lbs), Boissonnat et al., Dey et al., Niyogi et al.
The Reach

• Introduced by Herbert Federer (Curvature Measures 1959): classe of **sets with positive reach** allow to define curvature measures beyond smooth case.

• Used again in the context of **manifold reconstruction with topological guarantees** : Amenta et al. (lfs), Boissonnat et al., Dey et al., Niyogi et al.

• A set is **convex iff. its reach is infinite**
The Reach

For a compact set $K$ denote by:

- $rch\ K$ its reach,
- $rad\ K$ its radius, i.e. the radius of the smallest ball enclosing $K$

Then:

- If $rad\ K < rch\ K$ then $K$ is contractible,
- If $r < K$ then $rch\ (K \cap B(x, r)) \geq rch\ K$
Metric distortion

For a closed set $S \subset \mathbb{R}^d$, $d_S$ denotes the **geodesic distance** in $S$, i.e. $d_S(a, b)$ is the infimum of lengths of paths in $S$ between $a$ and $b$. 
Metric distortion

For a closed set $\mathcal{S} \subset \mathbb{R}^d$, $d_\mathcal{S}$ denotes the \textbf{geodesic distance} in $\mathcal{S}$, i.e. $d_\mathcal{S}(a, b)$ is the infimum of lengths of paths in $\mathcal{S}$ between $a$ and $b$.

\[ \forall a, b \in \mathcal{S}, d_\mathcal{S}(a, b) < \frac{\pi}{2} |a - b| \]

$\Rightarrow$ $\mathcal{S}$ is simply connected.

Metric distortion is another way to bound the size of topological features.
If $\mathbb{S}_r$ is a $(d - 1)$-sphere of radius $r$ in euclidean space $\mathbb{R}^d$, then $\text{rch} \mathbb{S}_r = r$ and:

$$\forall a, b \in \mathbb{S}_r, \quad d_{\mathbb{S}_r}(a, b) = 2r \arcsin \frac{|a - b|}{2r}$$
Metric distortion

**Theorem 1.** If $\mathcal{S} \subset \mathbb{R}^d$ is a closed set, then

$$\text{rch} \mathcal{S} = \sup \left\{ r > 0, \forall a, b \in \mathcal{S}, |a - b| < 2r \Rightarrow d_\mathcal{S}(a, b) \leq 2r \arcsin \frac{|a - b|}{2r} \right\},$$

where the sup over the empty set is 0.
Theorem 1. If $S \subset \mathbb{R}^d$ is a closed set, then

Corollary 2. Let $S \subset \mathbb{R}^d$ be a closed set with positive reach $r = \text{rch} \ S > 0$. Then, for any $r' < \text{rch} \ S$ and any $x \in \mathbb{R}^d$, if $B(x, r')$ is the closed ball centered at $x$ with radius $r'$, then $S \cap B(x, r')$ is geodesically convex in $S$. 
proof of geodesic convexity

\begin{theorem}
If \( S \subset \mathbb{R}^d \) is a closed set, then

\[ \text{rch} \, S = \sup \left\{ r > 0, \ \forall a, b \in S, \ |a - b| < 2r \Rightarrow d_S(a, b) \leq 2r \arcsin \frac{|a - b|}{2r} \right\}, \]

\end{theorem}

\begin{corollary}
Let \( S \subset \mathbb{R}^d \) be a closed set with positive reach \( r = \text{rch} \, S > 0 \). Then, for any \( r' < \text{rch} \, S \) and any \( x \in \mathbb{R}^d \), if \( B(x, r') \) is the closed ball centered at \( x \) with radius \( r' \), then \( S \cap B(x, r') \) is geodesically convex in \( S \).
\end{corollary}

For a contradiction assume a minimizing geodesic goes outside the ball with radius \( r' < \text{rch} \, S \):
proof of geodesic convexity

**Theorem 1.** If $S \subset \mathbb{R}^d$ is a closed set, then

$$\text{rch} S = \sup \left\{ r > 0, \forall a, b \in S, |a - b| < 2r \Rightarrow d_S(a, b) \leq 2r \arcsin \frac{|a - b|}{2r} \right\},$$

**Corollary 2.** Let $S \subset \mathbb{R}^d$ be a closed set with positive reach $r = \text{rch} S > 0$. Then, for any $r' < \text{rch} S$ and any $x \in \mathbb{R}^d$, if $B(x, r')$ is the closed ball centered at $x$ with radius $r'$, then $S \cap B(x, r')$ is geodesically convex in $S$.

For a contradiction assume a minimizing geodesic goes outside the ball with radius $r' < \text{rch} S$:

Focus on the path between $a'$ and $b'$. The projection on the sphere with radius $r'$ decreases lengths and:

$$d_S(a', b') > 2r' \arcsin \frac{|a' - b'|}{2r'} > 2r \arcsin \frac{|a' - b'|}{2r}$$

A contradiction with the theorem inequality.
Proof of Theorem 1

First the easy direction:

**Theorem 1.** If $\mathcal{S} \subset \mathbb{R}^d$ is a closed set, then

$$\mathrm{rch} \mathcal{S} = \sup \left\{ r > 0, \forall a, b \in \mathcal{S}, |a - b| < 2r \Rightarrow d_\mathcal{S}(a, b) \leq 2r \arcsin \frac{|a - b|}{2r} \right\},$$

where the sup over the empty set is 0.

If $\mathrm{rch} \mathcal{S} < r$ then there is $x$ in the medial axis with at least two points $a, b \in \mathcal{S}$ with $d(x, \mathcal{S}) = d(x, a) = d(x, b) = r' < r$ and:

$$|a - b| < 2r \quad \text{and} \quad d_\mathcal{S}(a, b) \geq 2r' \arcsin \frac{|a - b|}{2r'} > 2r \arcsin \frac{|a - b|}{2r}$$
Proof of Theorem 1

Now the less trivial direction:

**Theorem 1.** If $\mathcal{S} \subset \mathbb{R}^d$ is a closed set, then

$$\text{rch } \mathcal{S} = \sup \left\{ r > 0, \forall a, b \in \mathcal{S}, |a - b| < 2r \Rightarrow d_{\mathcal{S}}(a, b) \leq 2r \arcsin \frac{|a - b|}{2r} \right\},$$

where the sup over the empty set is 0.

**Lemma 6.** Let $\mathcal{S} \subset \mathbb{R}^d$ be a closed set with reach $r = \text{rch } \mathcal{S} > 0$. For any $a, b \in \mathcal{S}$ such that $\|a - b\| < 2r$ one has:

$$d_{\mathcal{S}}(a, b) \leq 2r \arcsin \frac{\|a - b\|}{2r}$$
Proof of Theorem 1

Now the less trivial direction:

Lemma 6. Let $S \subset \mathbb{R}^d$ be a closed set with reach $r = \text{rch} S > 0$. For any $a, b \in S$ such that $\|a - b\| < 2r$ one has:

$$d_S(a, b) \leq 2r \arcsin \frac{\|a - b\|}{2r}$$

We use two results from H. Federer:
Proof of Theorem 1

Now the less trivial direction:

We use two results from H. Federer:

1) Tubular neighborhood:

If $0 < d(x, S) < \text{rch } S$ and $\text{pr}(x)$ is the point in $S$ closest to $x$ then:

$$\forall \lambda \in [0, \text{rch } S), \text{pr} \left( \text{pr}(x) + \lambda \frac{x - \text{pr}(x)}{\|x - \text{pr}(x)\|} \right) = \text{pr}(x)$$
Proof of Theorem 1

Now the less trivial direction:

We use two results from H. Federer:

2) Projection is Lipschitz:

For $\mu < r = \text{rch} S$ the restriction of $\text{pr}$ to the $\mu$-tubular neighbourhood $\mathcal{S}^\mu$ is Lipschitz with constant:

$$\frac{\text{rch} S}{\text{rch} S - \mu}$$
Proof of Theorem 1

Now the less trivial direction:

Lemma 6. Let $S \subset \mathbb{R}^d$ be a closed set with reach $r = \text{rch} S > 0$. For any $a, b \in S$ such that $\|a - b\| < 2r$ one has:

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A first idea consists in applying the Lipschitz constant on the projection of the straight line:
Proof of Theorem 1

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A first idea consists in applying the Lipschitz constant on the projection of the straight line:

Unfortunately, even if this bounds the geodesic length, the bound is not tight.
Proof of Theorem 1

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$$d_S(a, b) \leq 2r \arcsin \frac{\|a - b\|}{2r}$$

But this works:

Step 0
Proof of Theorem 1

Now the less trivial direction:

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Now the less trivial direction:

Lemma 6. Let $S \subseteq \mathbb{R}^d$ be a closed set with reach $r = \operatorname{rch} S > 0$. For any $a, b \in S$ such that $\|a - b\| < 2r$ one has:

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But this works:

Step 2
Proof of Theorem 1

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Lemma 6. Let \( S \subseteq \mathbb{R}^d \) be a closed set with reach \( r = \text{rch} S > 0 \). For any \( a, b \in S \) such that \( \|a - b\| < 2r \) one has:

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\]

But this works:

Step 2

Medial axis

rch \( S \)

\( S \)

\( a \)

\( b \)
Proof of Theorem 1

Now the less trivial direction:

**Lemma 6.** Let $S \subset \mathbb{R}^d$ be a closed set with reach $r = \text{rch} S > 0$. For any $a, b \in S$ such that $\|a - b\| < 2r$ one has:

$$d_S(a, b) \leq 2r \arcsin \frac{\|a - b\|}{2r}$$

But this works:

Step 3
Proof of Theorem 1

Now the less trivial direction:

Lemma 6. Let $S \subset \mathbb{R}^d$ be a closed set with reach $r = \text{rch} S > 0$. For any $a, b \in S$ such that $\|a - b\| < 2r$ one has:

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But this works:

Step 3
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**Lemma 6.** Let $S \subset \mathbb{R}^d$ be a closed set with reach $r = \text{rch} S > 0$. For any $a, b \in S$ such that $\|a - b\| < 2r$ one has:

$$d_S(a, b) \leq 2r \arcsin \frac{\|a - b\|}{2r}$$

But this works:

Step 4

![Diagram of the medial axis and reach of a set](image)
Proof of Theorem 1

Now the less trivial direction:

Lemma 6. Let $S \subset \mathbb{R}^d$ be a closed set with reach $r = \text{rch} S > 0$. For any $a, b \in S$ such that $\|a - b\| < 2r$ one has:

$$d_S(a, b) \leq 2r \arcsin \frac{\|a - b\|}{2r}$$

But this works:

Step 4
Proof of Theorem 1

Now the less trivial direction:

**Lemma 5.** Let $S \subset \mathbb{R}^d$ be a closed set with reach $r = \text{rch} S > 0$. For $a, b \in S$ such that $\delta = \frac{||a-b||}{2} < r$ and $m = \frac{a+b}{2}$ one has:

$$||\pi_S(m) - m|| \leq \rho$$

with:

$$\rho = r - \sqrt{r^2 - \delta^2}$$

**Figure 1** On the left the projection $\pi_S(m)$ is contained in the disk of center $m$ and radius $\rho$. The notation used in the proof of Lemma 3 is also added. From the right figure it is easy to deduce that $\rho = r - \sqrt{r^2 - \delta^2}$. 
Proof of Theorem 1

Now the less trivial direction:

Lemma 6. Let $S \subset \mathbb{R}^d$ be a closed set with reach $r = \text{rch} S > 0$. For any $a, b \in S$ such that $\|a - b\| < 2r$ one has:

$$d_S(a, b) \leq 2r \arcsin \frac{\|a - b\|}{2r}$$

$$\text{length}(\phi_i) = \sum_{k=0}^{2^i-1} |\phi_i((k + 1)/2^i) - \phi_i(k/2^i)|$$

$$\leq \sum_{k=0}^{2^i-1} |\tilde{\phi_i}((k + 1)/2^i) - \tilde{\phi_i}(k/2^i)|$$

$$= \text{length}(\tilde{\phi_i}) \leq \text{length}(\tilde{C}_{\tilde{a}, \tilde{b}}) = 2r \arcsin \frac{|a - b|}{2r}.$$
Proof of Theorem 1

Now the less trivial direction:

**Lemma 6.** Let $S \subset \mathbb{R}^d$ be a closed set with reach $r = \text{rch} S > 0$. For any $a, b \in S$ such that $\|a - b\| < 2r$ one has:

$$d_S(a, b) \leq 2r \arcsin \frac{\|a - b\|}{2r}$$

\[
\delta_i = \frac{1}{2} \max_{0 \leq k \leq 2^i - 1} |\phi_i((k + 1)/2^i) - \phi_i(k/2^i)|. \quad \lim_{i \to \infty} \delta_i = 0.
\]

\[
\text{length}(\pi_S \circ \phi_i) \leq rch S \frac{\text{length}(\phi_i)}{rch S - \delta_i} \leq rch S \frac{2r \arcsin |a - b|}{rch S - \delta_i}
\]
Embedded manifolds with positive reach

If $\mathcal{M}$ is a $C^{1,1}$ compact manifold embedded in $\mathbb{R}^d$ then $rch \mathcal{M} > 0$.

If $\mathcal{M}$ is a manifold embedded in $\mathbb{R}^d$ with $rch \mathcal{M} > 0$ then $\mathcal{M}$ is $C^{1,1}$.
Reach and curvature

If $\mathcal{M}$ is a $C^2$ manifold embedded in $\mathbb{R}^d$, and $\Pi_p$ denotes its second fundamental form at point $p \in \mathcal{M}$, then:

$$||\Pi_p|| = \sup_{||u||=||v||=1} ||\Pi_p(u,v)|| \leq \sup_{||w||=1} ||\Pi_p(w,w)|| \leq \frac{1}{rch \mathcal{M}}$$

**Lemma 9.** Let $\gamma(t)$ be a geodesic parametrized according to arc length on $\mathcal{M} \subset \mathbb{R}^d$, then $|\dddot{\gamma}| \leq 1/rch(\mathcal{M})$, where we use Newton’s notation, that is we write $\dddot{\gamma}$ for the second derivative of $\gamma$ with respect to $t$. 
Reach and curvature

If \( \mathcal{M} \) is a \( C^2 \) manifold embedded in \( \mathbb{R}^d \), and \( \Pi_p \) denotes its second fundamental form at point \( p \in \mathcal{M} \), then:

\[
\|\Pi_p\| = \sup_{\|u\|=\|v\|=1} \|\Pi_p(u, v)\| \leq \sup_{\|w\|=1} \|\Pi_p(w, w)\| \leq \frac{1}{\text{rch } \mathcal{M}}
\]

Lemma 9. Let \( \gamma(t) \) be a geodesic parametrized according to arc length on \( \mathcal{M} \subset \mathbb{R}^d \), then

\[
|\dddot{\gamma}| \leq 1/\text{rch}(\mathcal{M}),
\]

where we use Newton’s notation, that is we write \( \dddot{\gamma} \) for the second derivative of \( \gamma \) with respect to \( t \).

\[
\Pi_p(w, w) = \Pi_p(\dot{\gamma}_w, \dot{\gamma}_w) = \nabla_{\dot{\gamma}_w} \dot{\gamma}_w - \nabla_{\dot{\gamma}_w} \dot{\gamma}_w = \nabla_{\dot{\gamma}_w} \dot{\gamma}_w - 0 = \ddot{\gamma}_w
\]
If $\mathcal{M}$ is a $C^2$ manifold embedded in $\mathbb{R}^d$, then:

**Lemma 11.** Let $p, q \in \mathcal{M}$, then

$$\angle(T_p \mathcal{M}, T_q \mathcal{M}) \leq \frac{d_\mathcal{M}(p, q)}{\text{rch}(\mathcal{M})}.$$

**Lemma 6.**

$$d_\mathcal{S}(a, b) \leq 2r \arcsin \frac{\|a - b\|}{2r}.$$
Lemma 11. Let $p, q \in \mathcal{M}$, then

$$\angle(T_p \mathcal{M}, T_q \mathcal{M}) \leq \frac{d_{\mathcal{M}}(p, q)}{\text{rch}(\mathcal{M})}.$$ 

By definition:

$$\angle(T_p \mathcal{M}, T_q \mathcal{M}) = \sup_{u \in T_p \mathcal{M}} \inf_{v \in T_q \mathcal{M}} \angle u, v$$

And therefore if $d_{\mathcal{M}}(p, q) = l$ and $\gamma$ is a geodesic parametrized by arc length such that $\gamma(0) = p$ and $\gamma(l) = q$,

if $t \mapsto u(t)$ is the parallel transport of a unit vector $u = u(0)$ along $\gamma$ then:

$$\angle(T_p \mathcal{M}, T_q \mathcal{M}) \leq \sup_{u \in T_p \mathcal{M}} \angle u(0), u(l) \leq \int_0^l \frac{du}{dt} dt$$

$\gamma(0) = p$

$\gamma(l) = q$
Lemma 11. Let \( p, q \in \mathcal{M} \), then

\[
\angle(T_p\mathcal{M}, T_q\mathcal{M}) \leq \frac{d_{\mathcal{M}}(p, q)}{\text{rch}(\mathcal{M})}.
\]

if \( t \mapsto u(t) \) is the parallel transport of a unit vector \( u = u(0) \) along \( \gamma \) then:

\[
\angle(T_p\mathcal{M}, T_q\mathcal{M}) \leq \sup_{u \in T_p\mathcal{M}} \angle u(0), u(l) \leq \int_{0}^{l} \frac{du}{dt} dt
\]

\[
\frac{du}{dt} = \bar{\nabla}_{\dot{\gamma}} u(t) = \Pi_{\gamma(t)}(\dot{\gamma}, u(t)) + \nabla_{\dot{\gamma}} u(t) = \Pi_{p}(\dot{\gamma}, u(t))
\]
Tangent variation on Manifolds

Lemma 11. Let $p, q \in \mathcal{M}$, then

$$\angle(T_p\mathcal{M}, T_q\mathcal{M}) \leq \frac{d_{\mathcal{M}}(p, q)}{\text{rch} (\mathcal{M})}.$$  

if $t \mapsto u(t)$ is the parallel transport of a unit vector $u = u(0)$ along $\gamma$ then:

$$\angle(T_p\mathcal{M}, T_q\mathcal{M}) \leq \sup_{u \in T_p\mathcal{M}} \angle u(0), u(l) \leq \int_0^l \frac{du}{dt} \, dt$$

$$\left\| \frac{du}{dt} \right\| = \| \tilde{\nabla}_{\dot{\gamma}} u(t) \| = \| \Pi_{\gamma(t)}(\dot{\gamma}, u(t)) + \nabla_{\dot{\gamma}} u(t) \| = \| \Pi_p(\dot{\gamma}, u(t)) \| \leq \frac{1}{\text{rch} \mathcal{M}}$$
A pictorial proof
Thank you
Medial Axis and Reach

Medial axis of an open set $O$:
« set of points in $O$ who have at least two closest points on the boundary of $O$ »

Medial axis of a closed set $C$
= Medial axis of complement of $C$:
« set of points who have at least two closest points in $C$ »

Reach of a closed set $C$
« infimum of distances between $C$ and its medial axis»