Reach, metric distorsion and variation of tangent space

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Join work with
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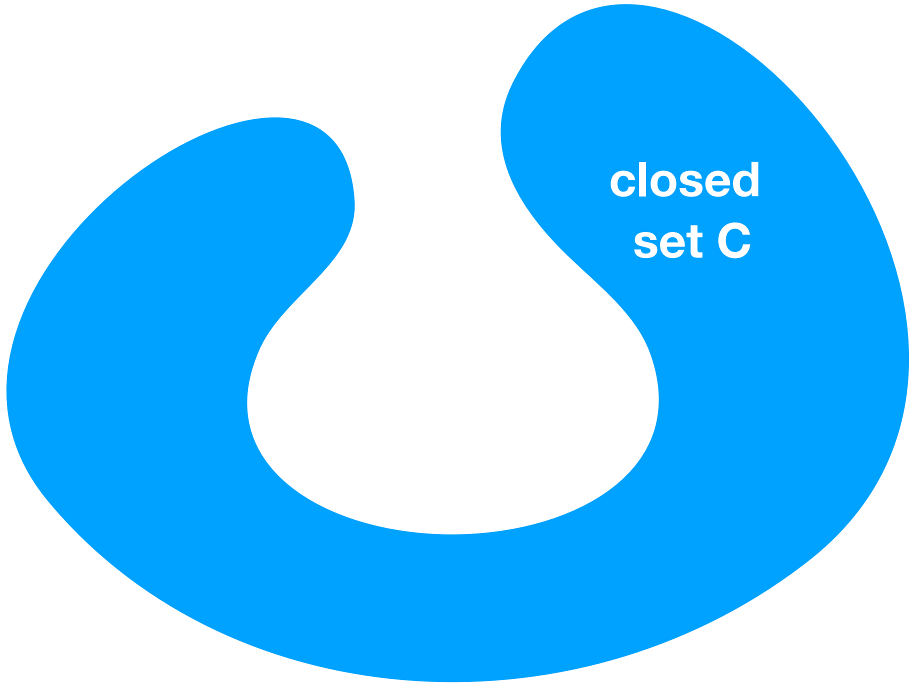
and Mathijs Wintraecken

Motivation

- Reach,
- metric distorsion,
- variation of tangent space

These are general geometric properties encountered in the proofs of several theorems that state **topological faithful reconstructions** of **manifolds** as well as **more general subsets of Euclidean space** by **simplicial complexes**

Medial Axis and Reach



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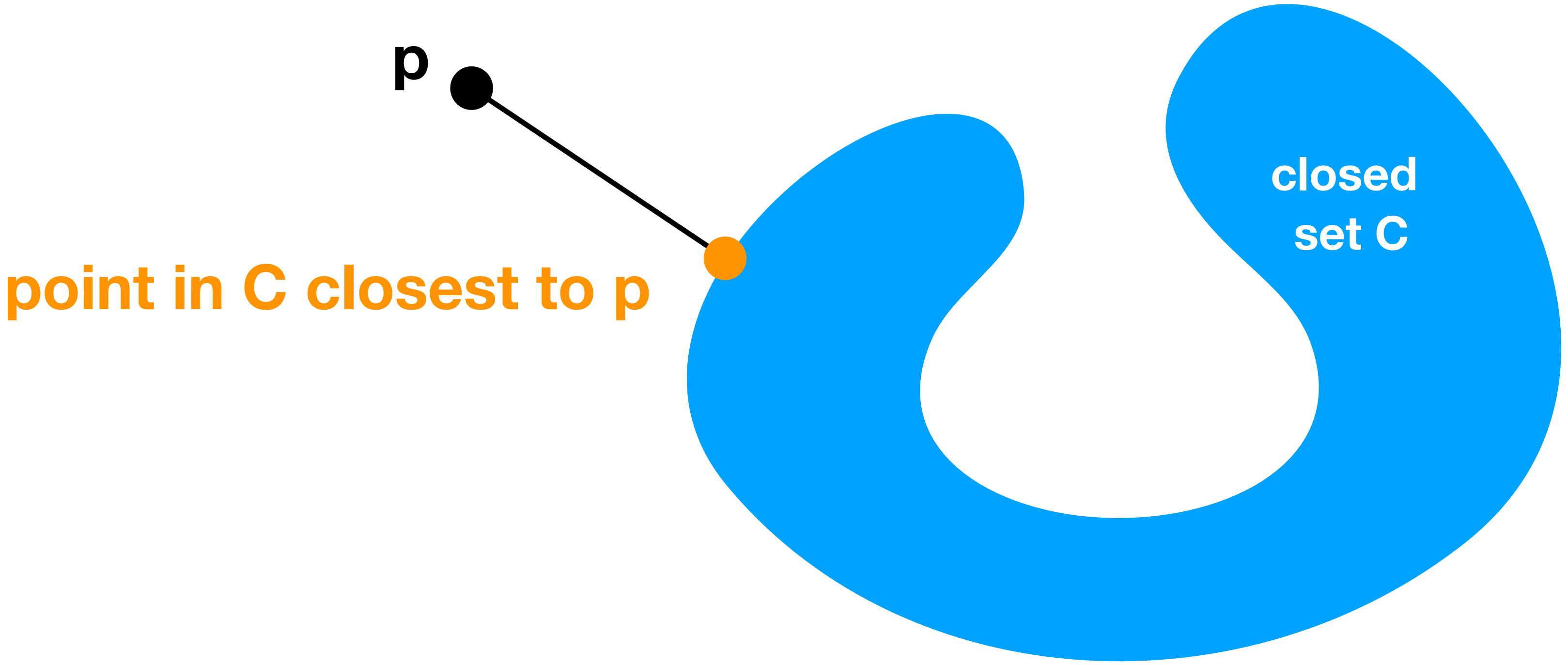

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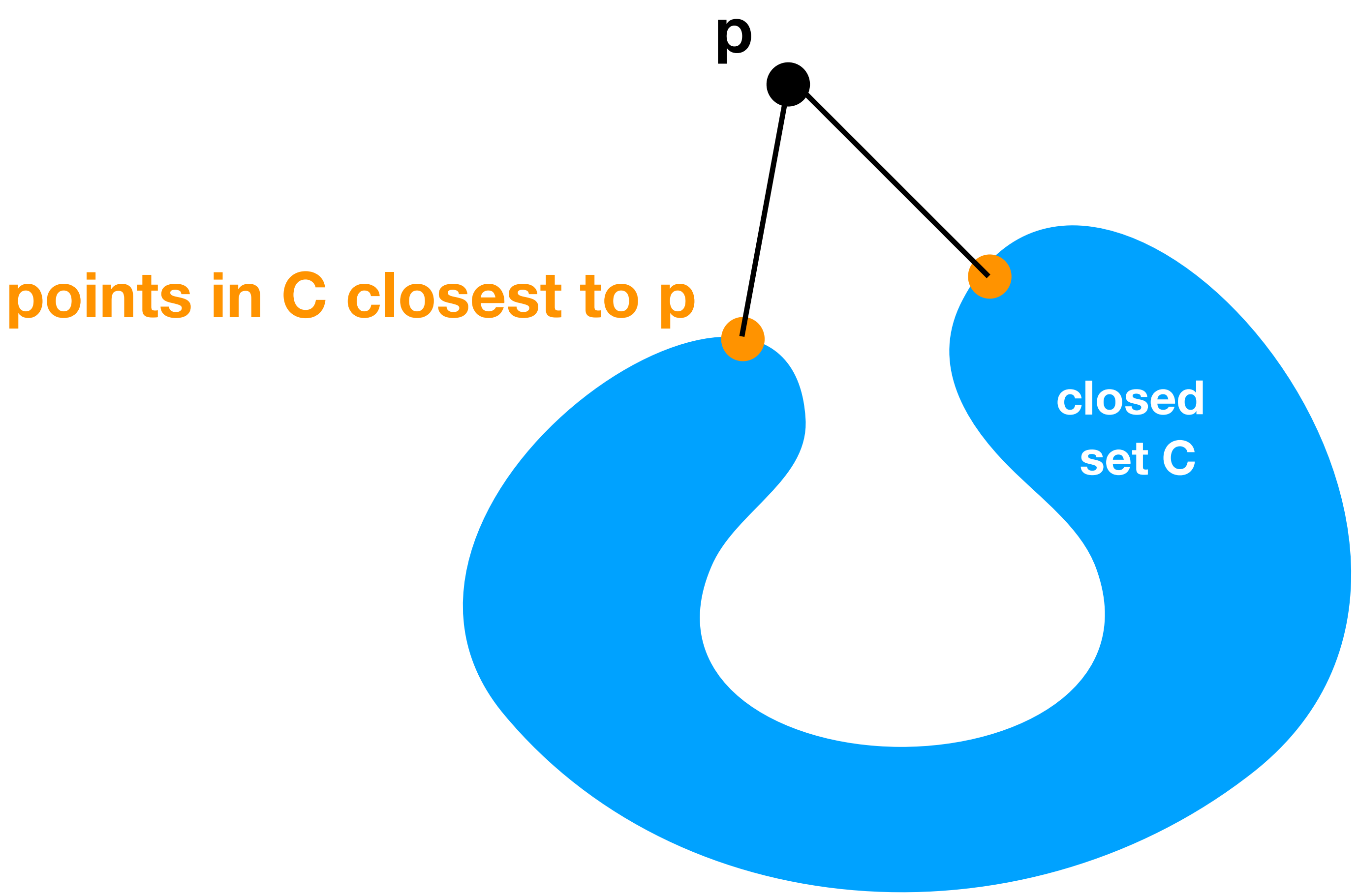

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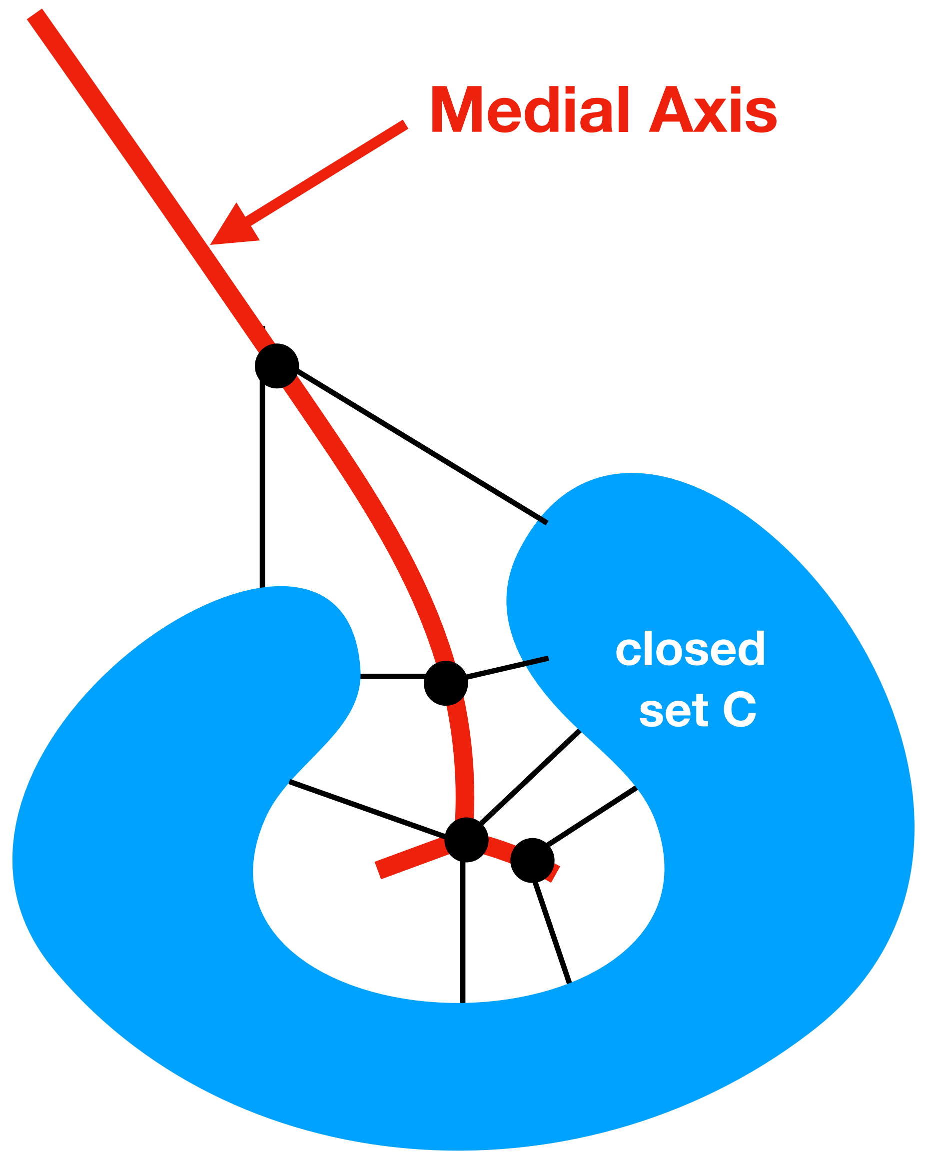

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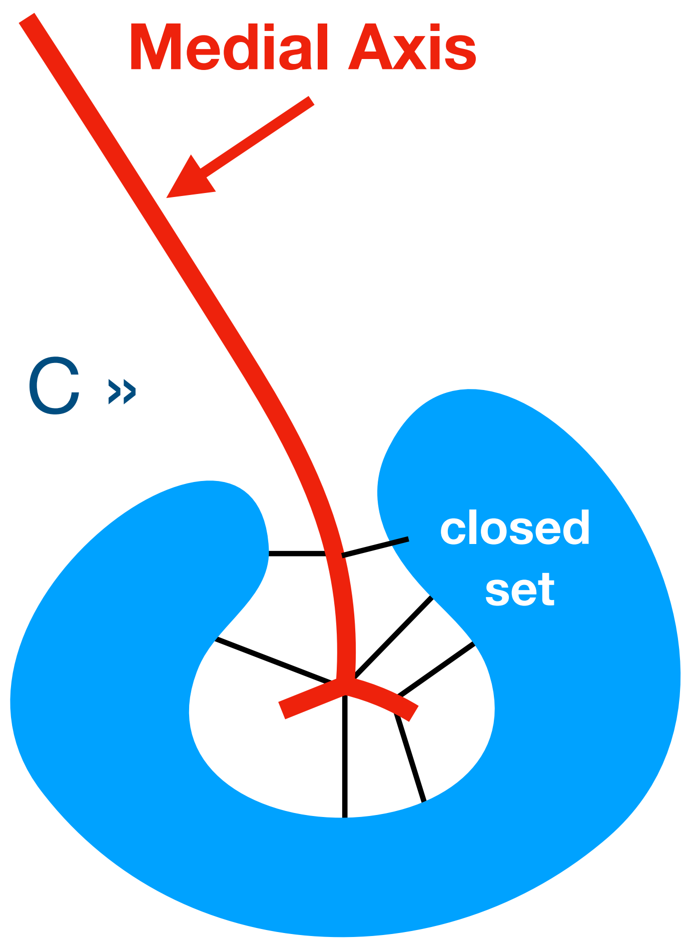

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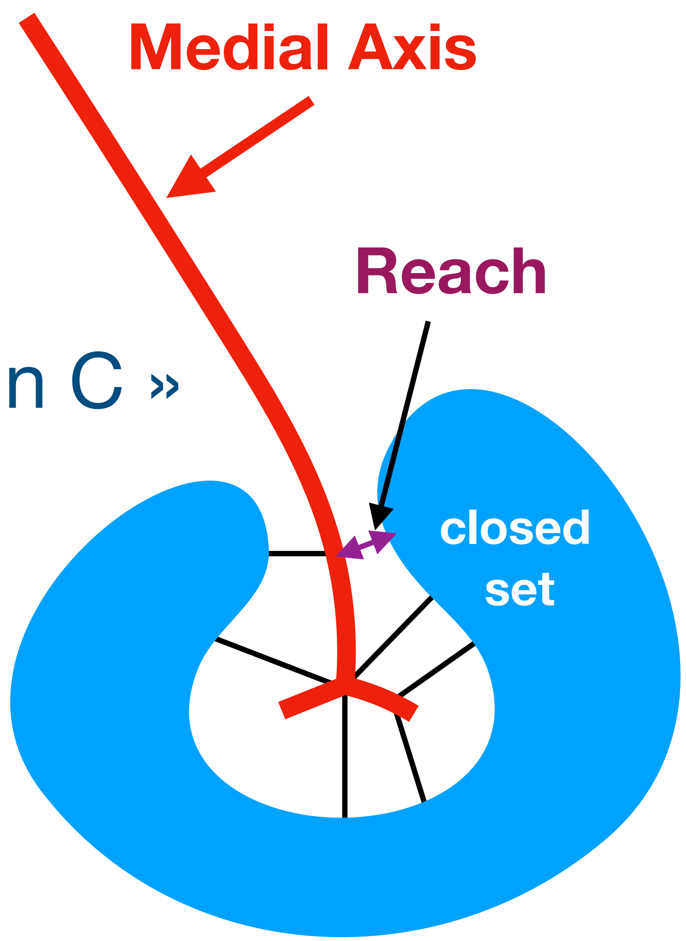

Medial axis of a **closed set C is**
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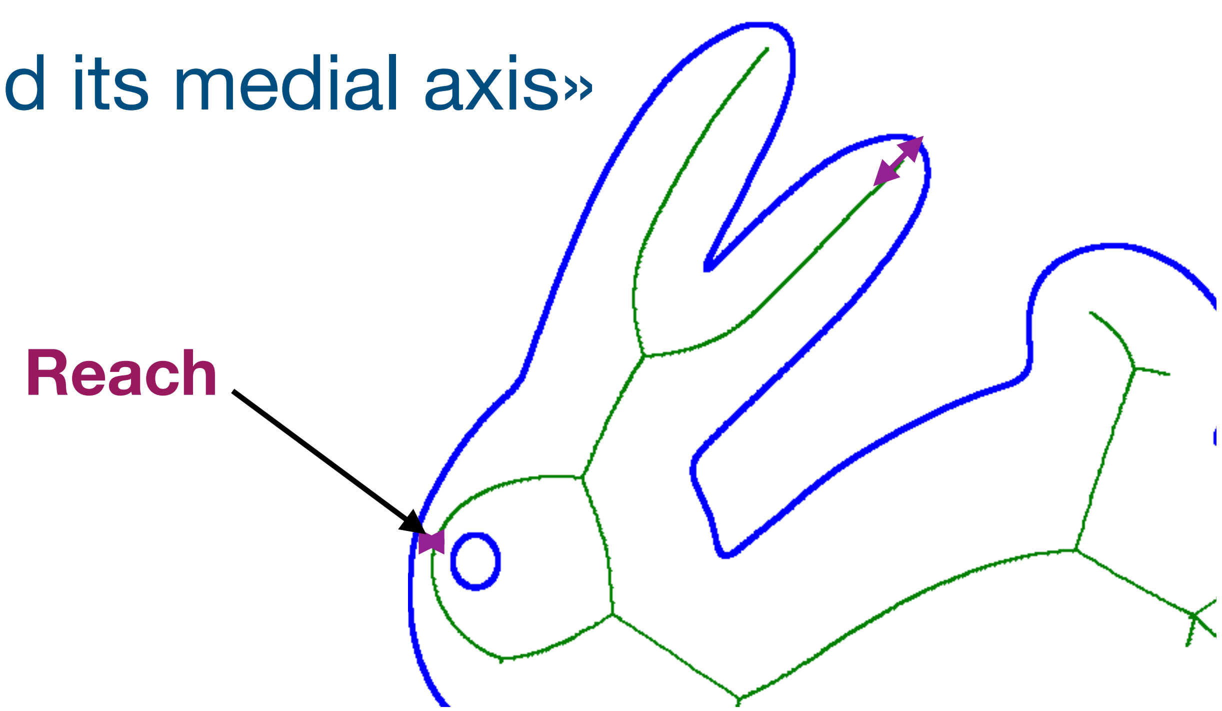
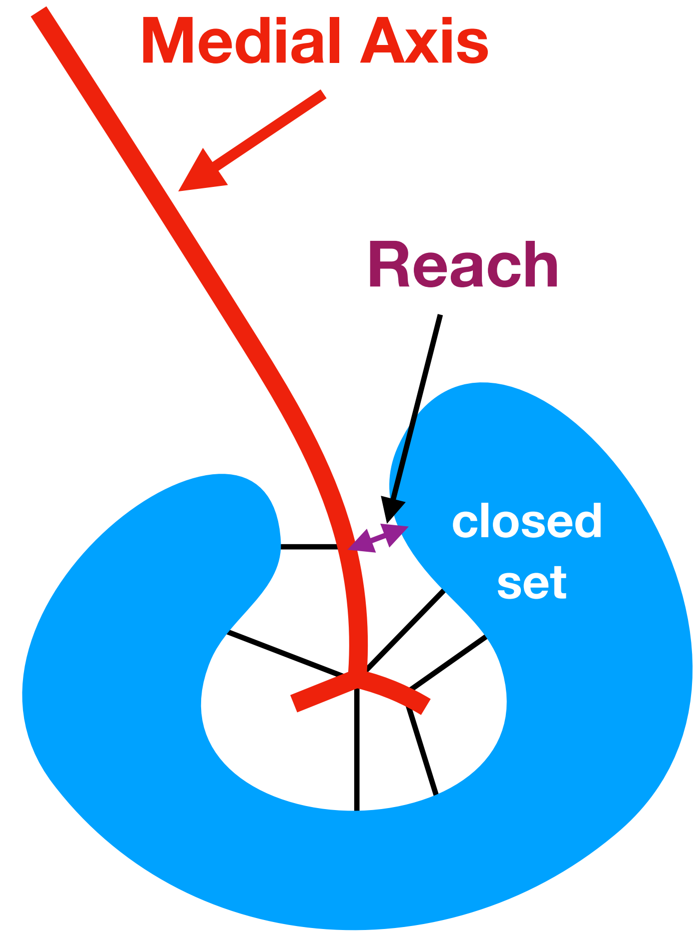
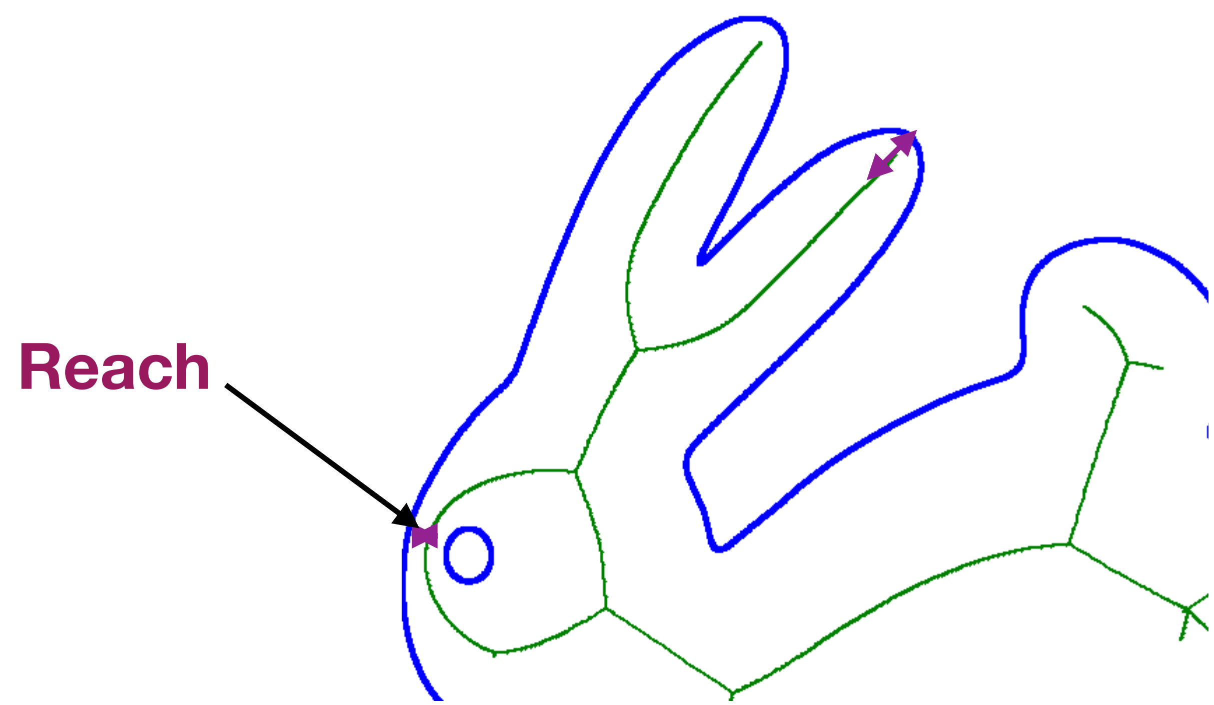
Reach of a **closed set C**
« infimum of distances between C and its medial axis»



The Reach

- Introduced by **Herbert Federer** (Curvature Measures 1959): classe of **sets with positive reach** allow to define curvature measures beyond smooth case.





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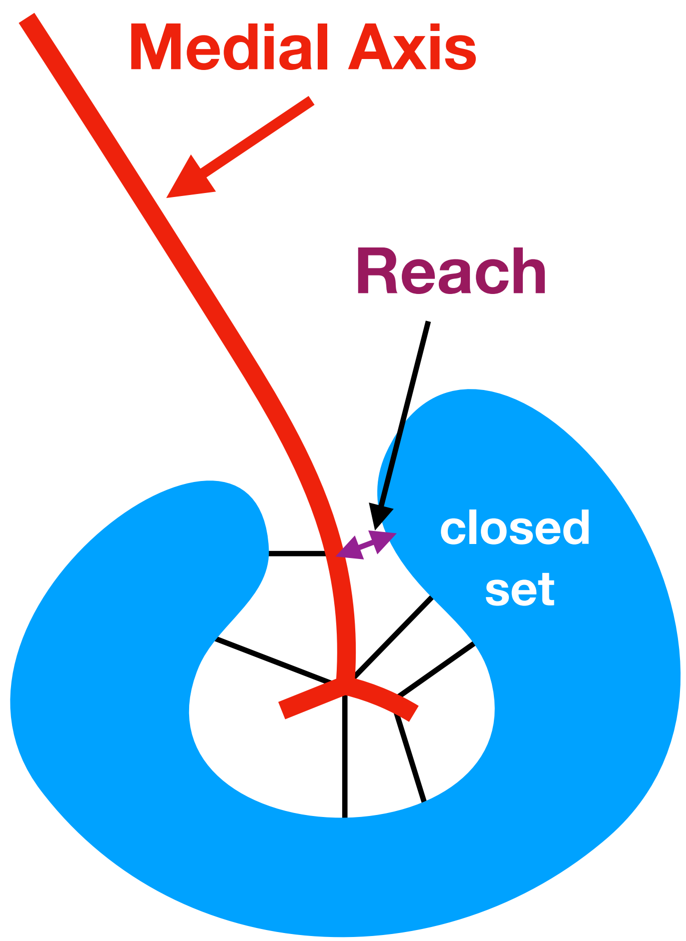

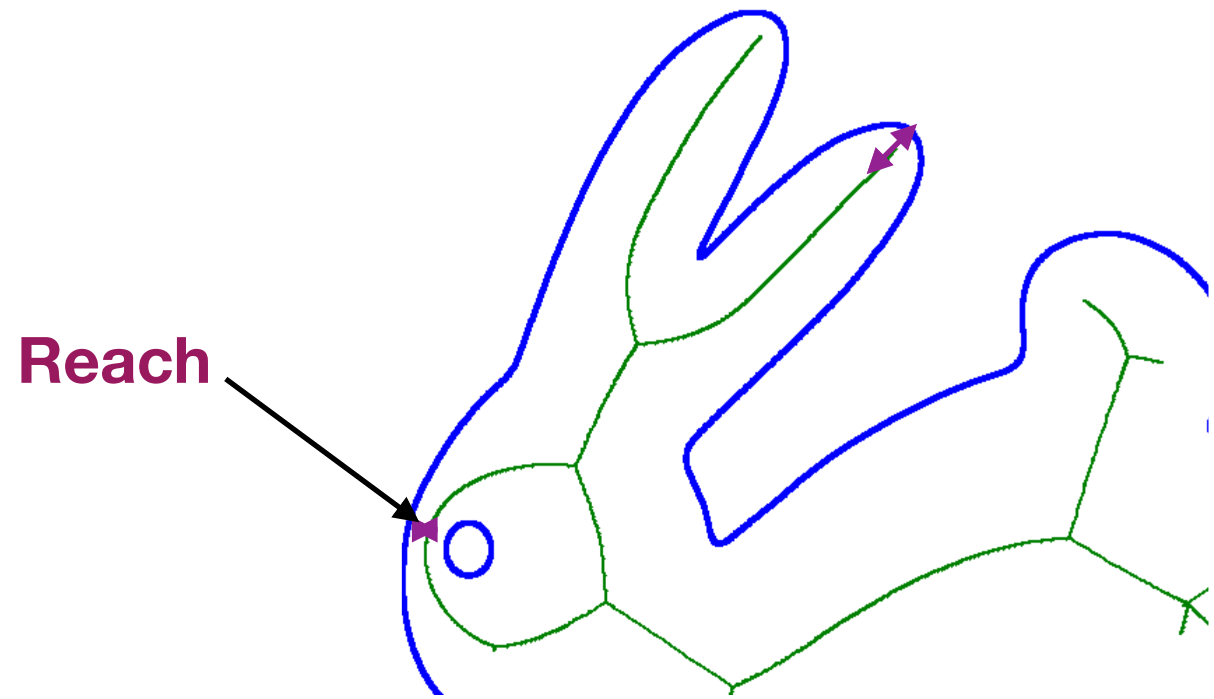

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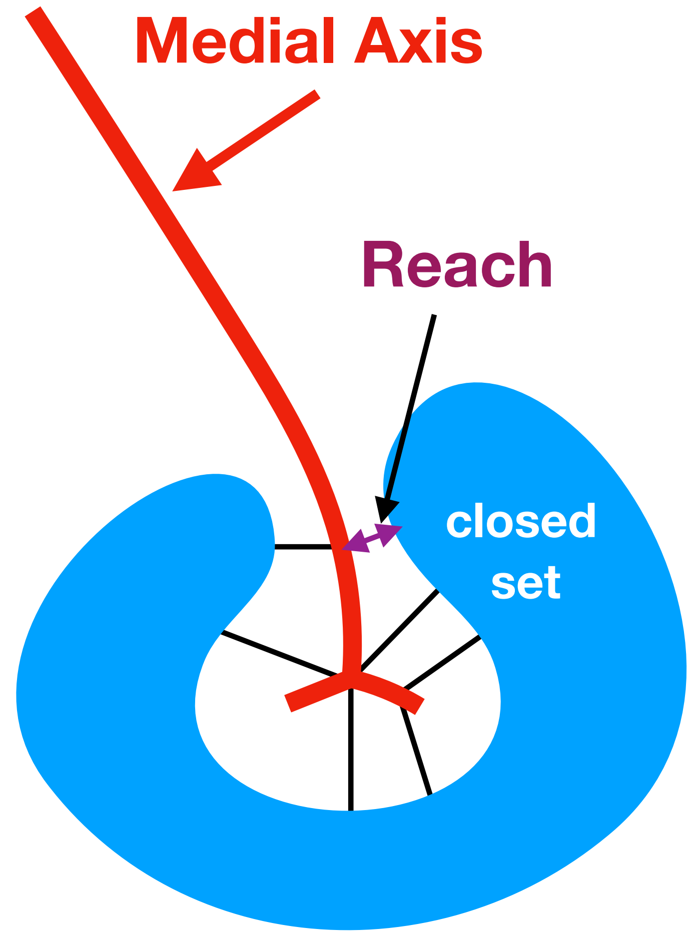

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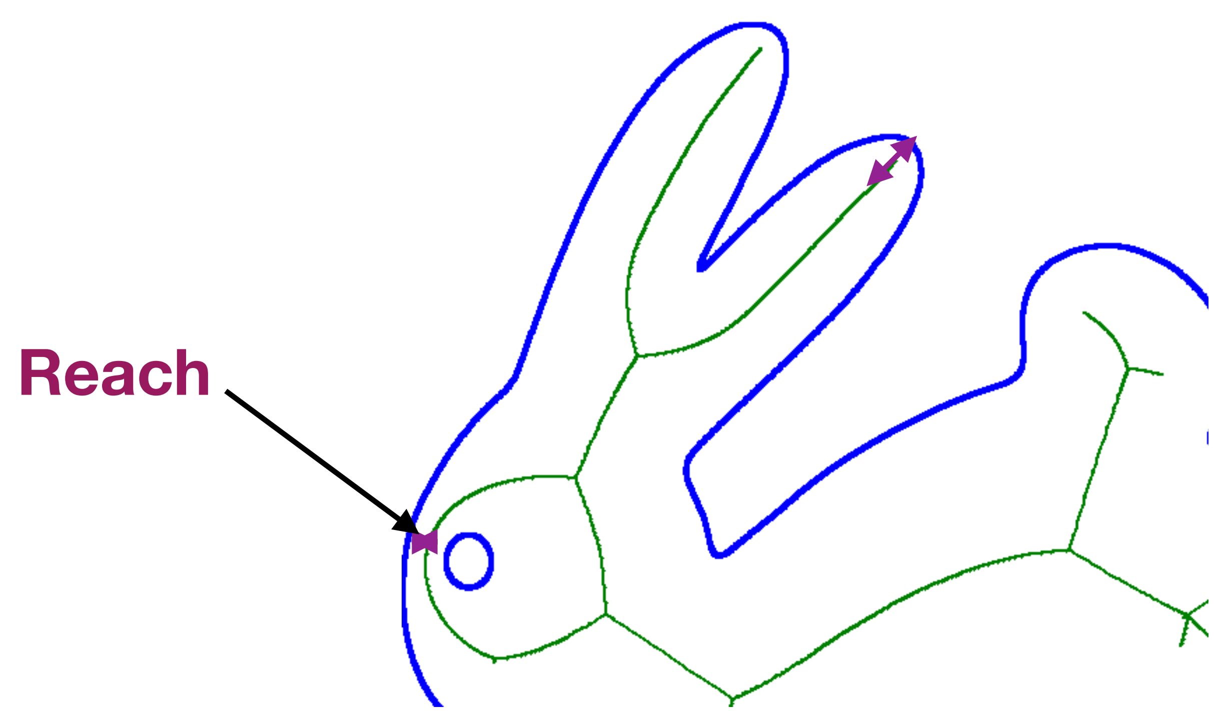
- A set is **convex iff**. its **reach is infinite**



The Reach

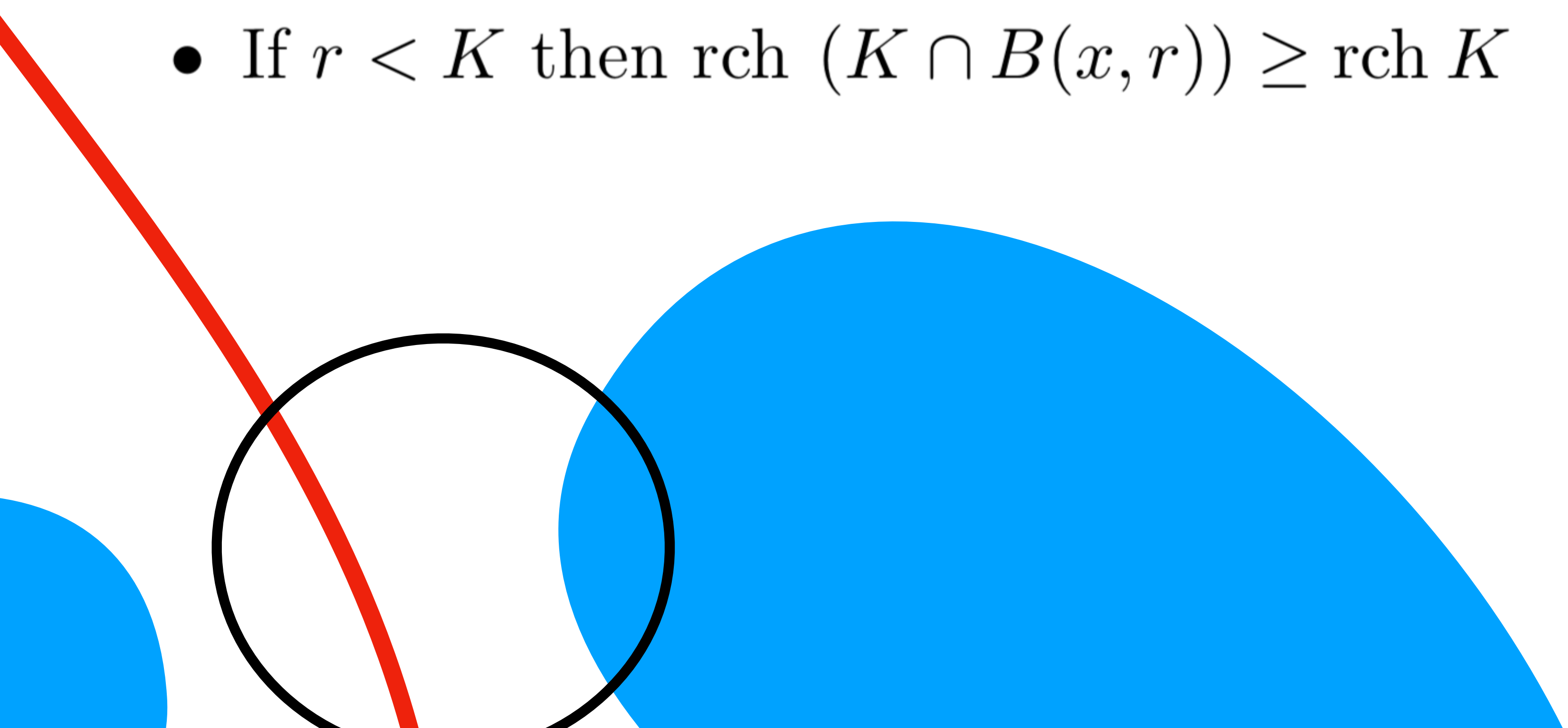
For a compact set K denote by:

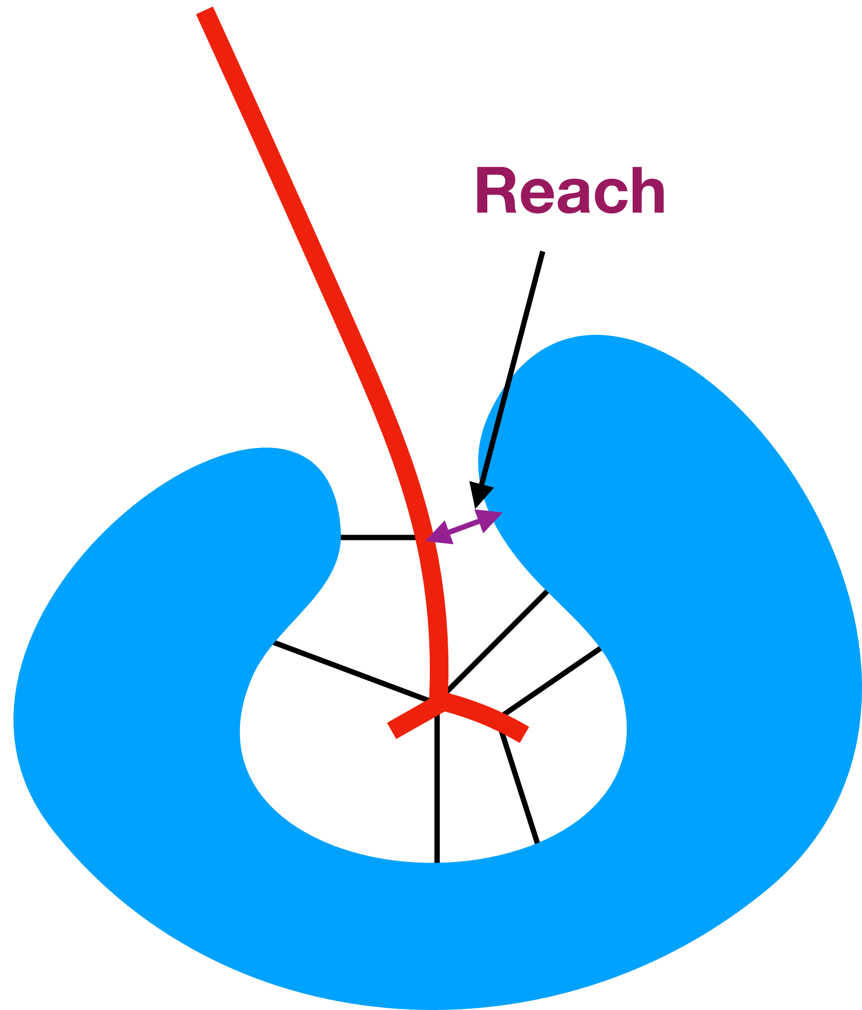
The reach is one way (among others) to bound the size of topological features.

- $\mathrm{rch}\, K$ its **reach**,
- $\mathrm{rad}\, K$ its **radius**, i.e. the radius of the smallest ball enclosing K

Then:

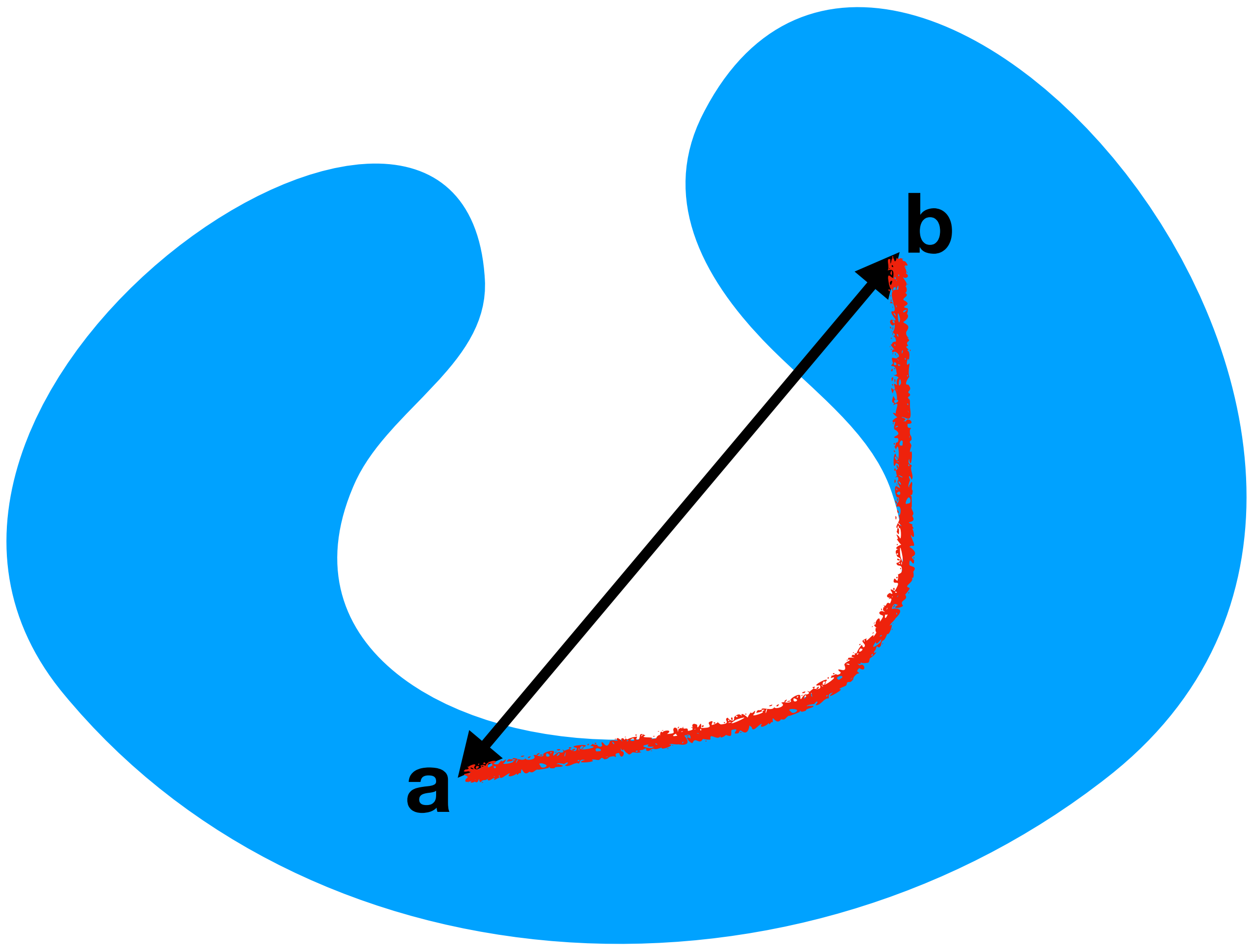
- If $\mathrm{rad}\, K < \mathrm{rch} K$ then K is contractible,
- If $r < K$ then $\mathrm{rch}\ (K \cap B(x, r)) \geq \mathrm{rch}\ K$





Metric distorsion

For a closed set $\mathcal{S} \subset \mathbb{R}^d$, $d_{\mathcal{S}}$ denotes the **geodesic distance** in $\mathcal{S}$, i.e. $d_{\mathcal{S}}(a, b)$ is the infimum of lengths of paths in $\mathcal{S}$ between a and b.



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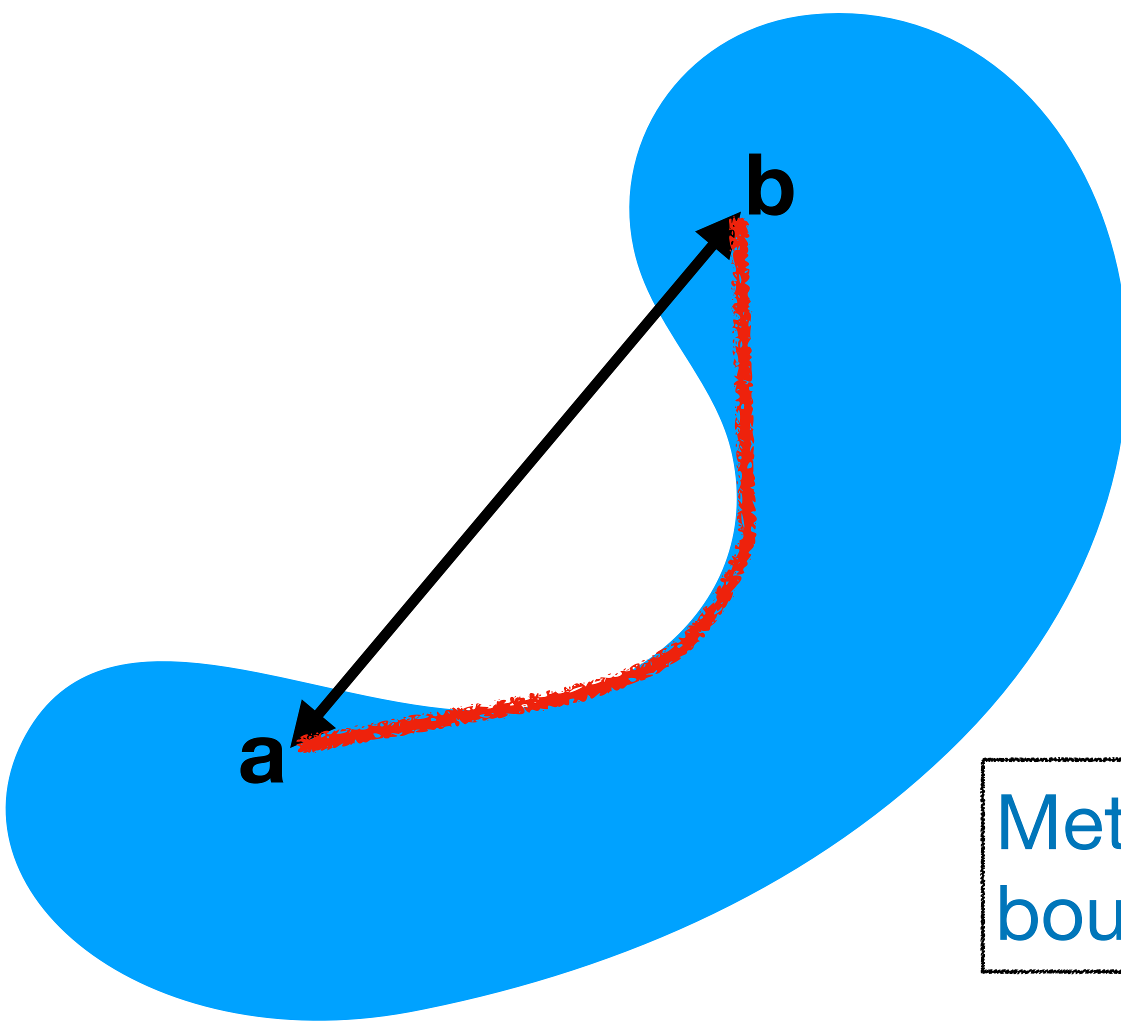


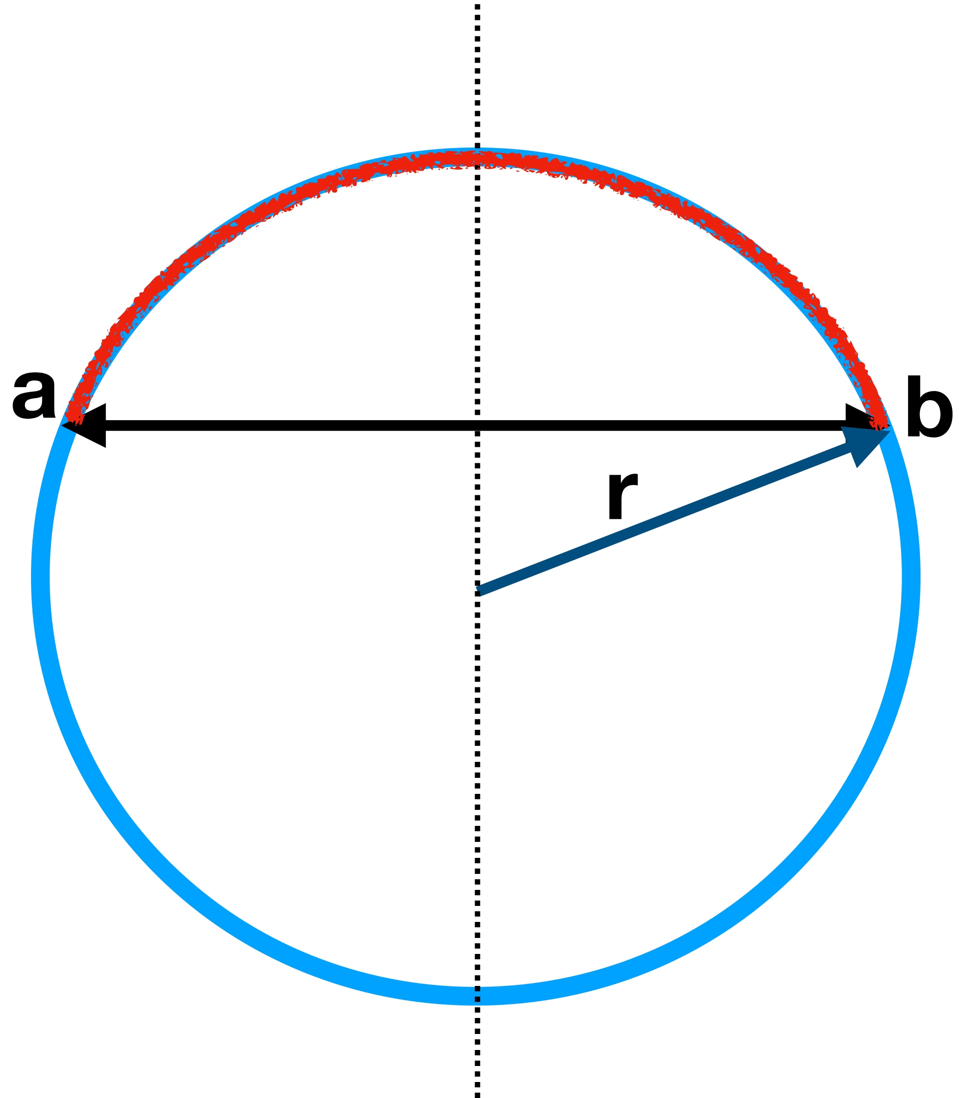
$$\forall a, b \in \mathcal{S}, d_{\mathcal{S}}(a, b) < \frac{\pi}{2}|a - b|$$
$$\Rightarrow \quad \mathcal{S} \quad \text{is simply connected}$$

Metic distorsion is another way to bound the size of topological features.

Metric distorsion

If $\mathbb{S}_r$ is a $(d-1)$-sphere of radius r in euclidean space $\mathbb{R}^d$, then $\operatorname{rch} \mathbb{S}_r = r$ and:

$$\forall a, b \in \mathbb{S}_r, \quad d_{\mathbb{S}_r}(a,b) = 2r \arcsin \frac{|a-b|}{2r}$$

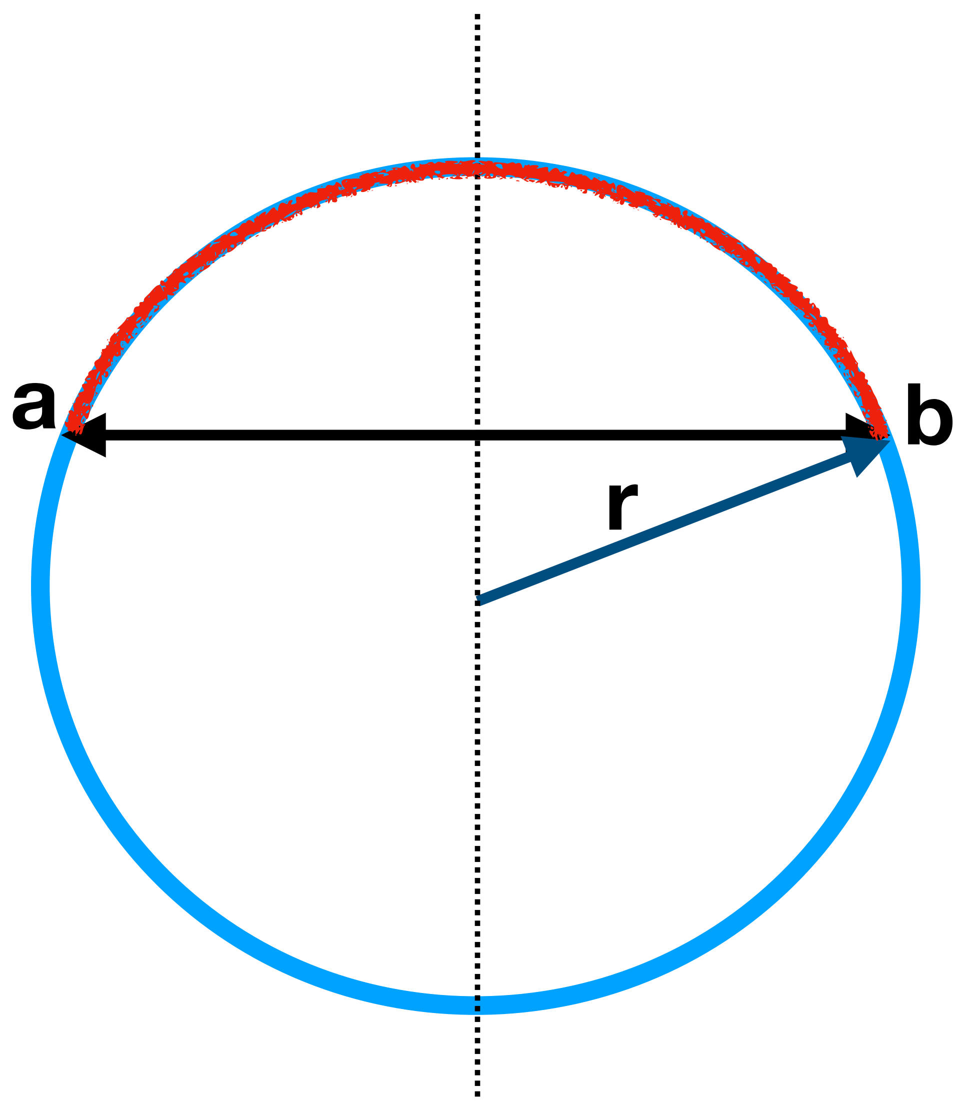


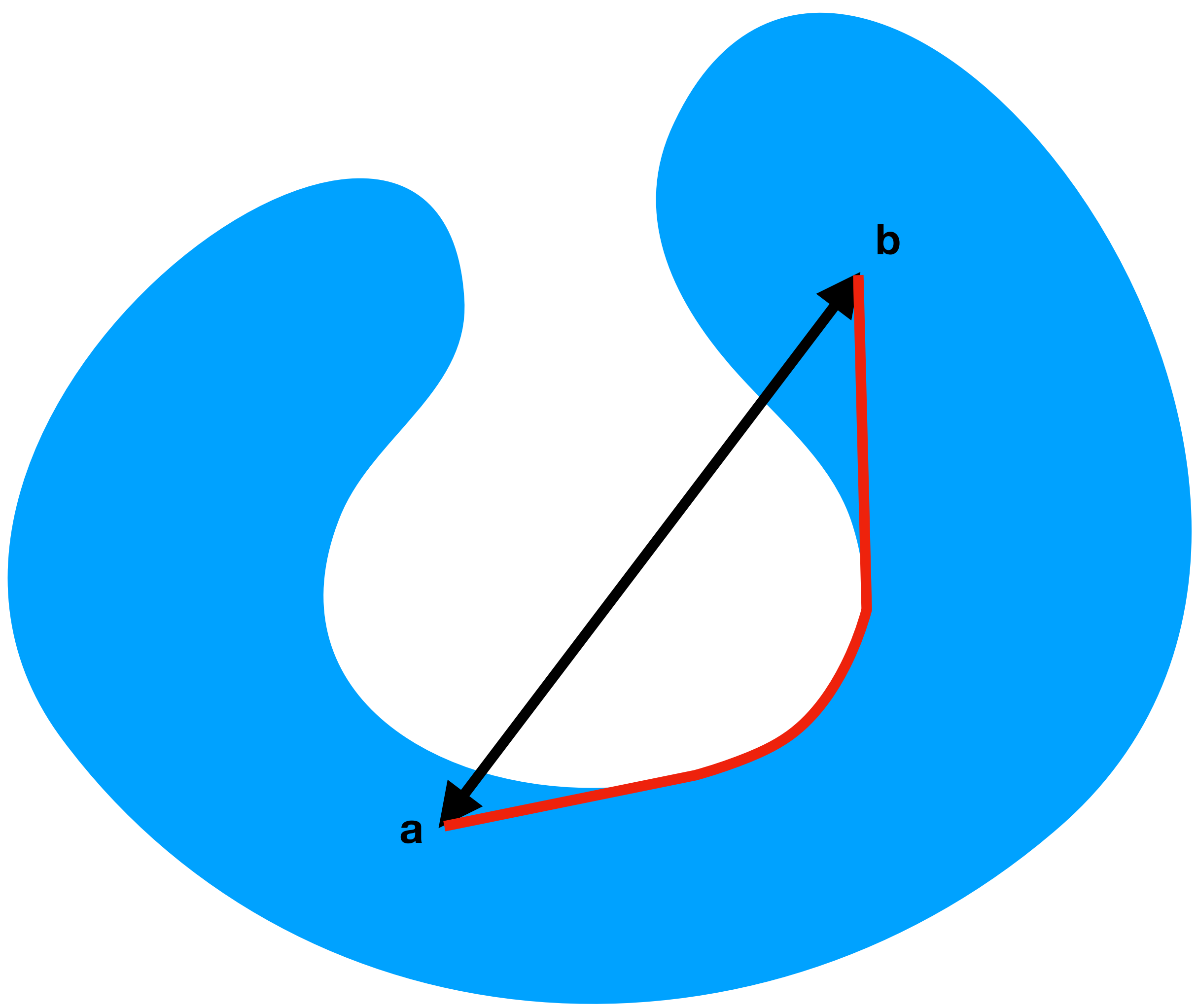
Metric distorsion

Theorem 1. *If $\mathcal{S} \subset \mathbb{R}^d$ is a closed set, then*

$$\operatorname{rch} \mathcal{S} = \sup \left\{ r > 0,\ \forall a, b \in \mathcal{S},\ |a-b| < 2r \Rightarrow d_{\mathcal{S}}(a,b) \le 2r \arcsin \frac{|a-b|}{2r} \right\},$$

where the sup *over the empty set is* 0.

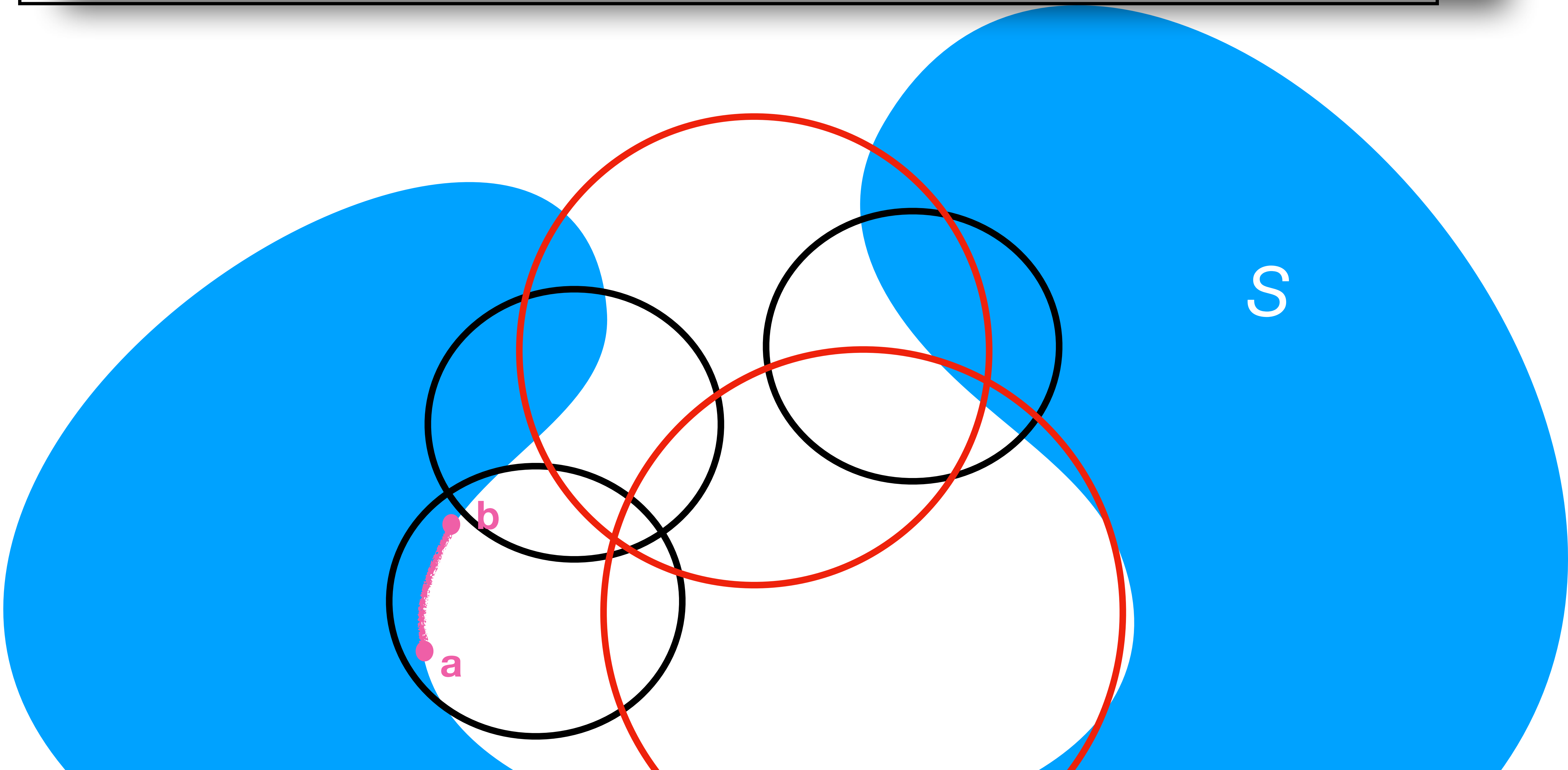




Corollary: geodesic convexity

Theorem 1. *If $\mathcal{S} \subset \mathbb{R}^d$ is a closed set, then*

Corollary 2. *Let $\mathcal{S} \subset \mathbb{R}^d$ be a closed set with positive reach $r = \operatorname{rch} \mathcal{S} > 0$. Then, for any $r' < \operatorname{rch} \mathcal{S}$ and any $x \in \mathbb{R}^d$, if $B(x, r')$ is the closed ball centered at x with radius r', then $\mathcal{S} \cap B(x, r')$ is geodesically convex in $\mathcal{S}$.*



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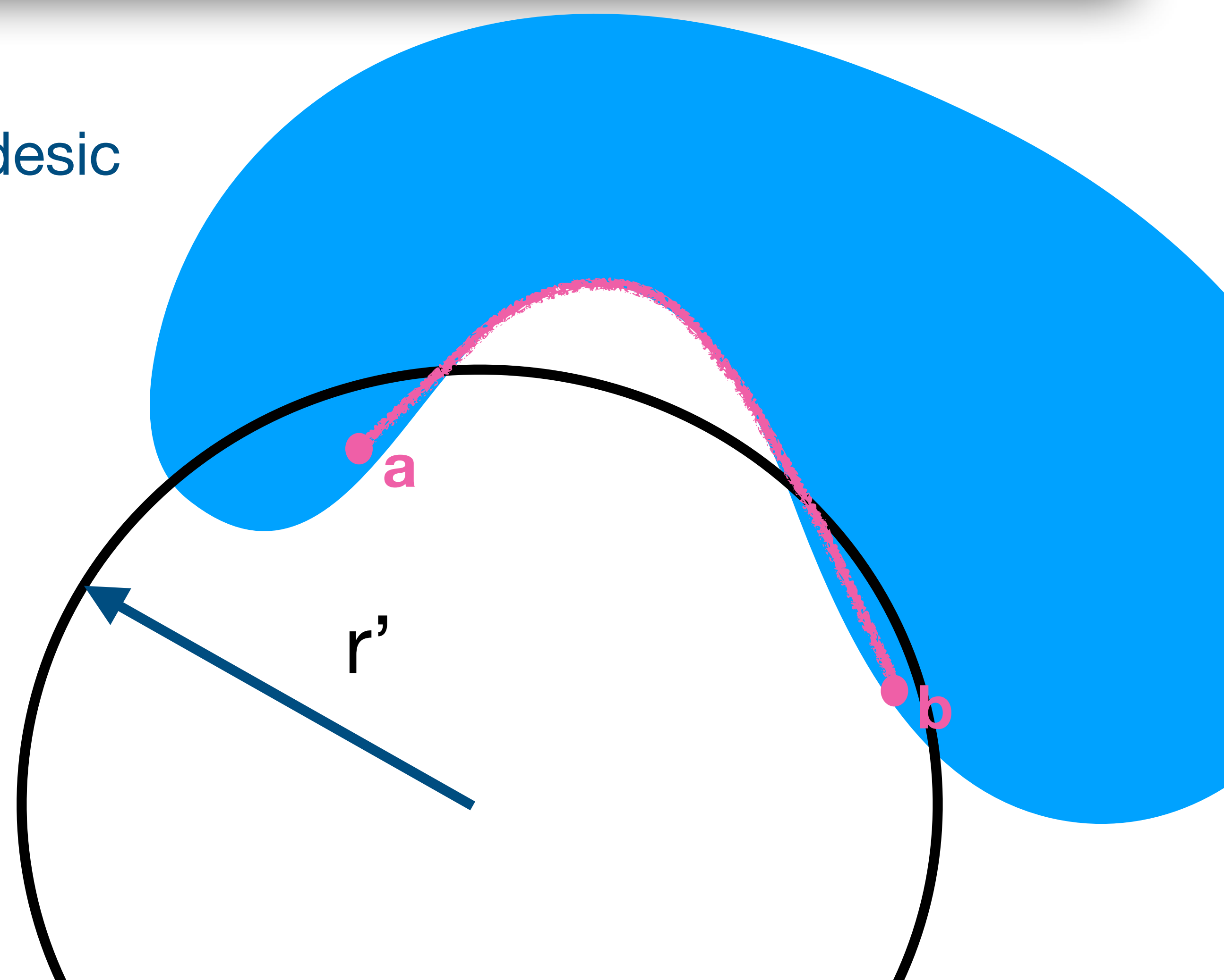
For a contradiction assume a minimizing geodesic goes outside the ball with radius r’ < rch *S*:



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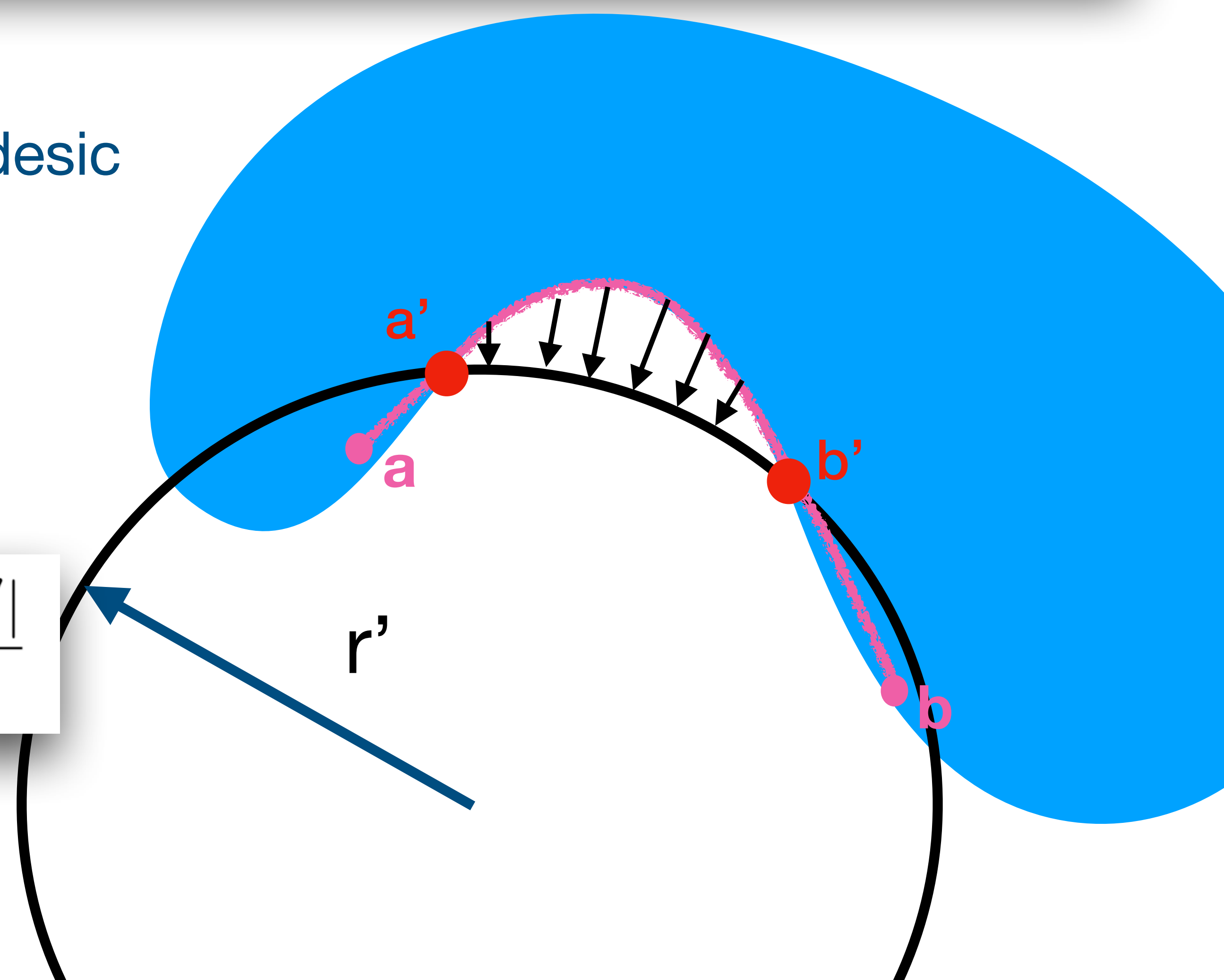
Focus on the path between a' and b'. The projection on the sphere with radius r' decreases lengths and:

$$d_{\mathcal{S}}(a', b') > 2r' \arcsin \frac{|a'-b'|}{2r'} > 2r \arcsin \frac{|a'-b'|}{2r}$$

A contradiction with the theorem inequality.



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First the easy direction:

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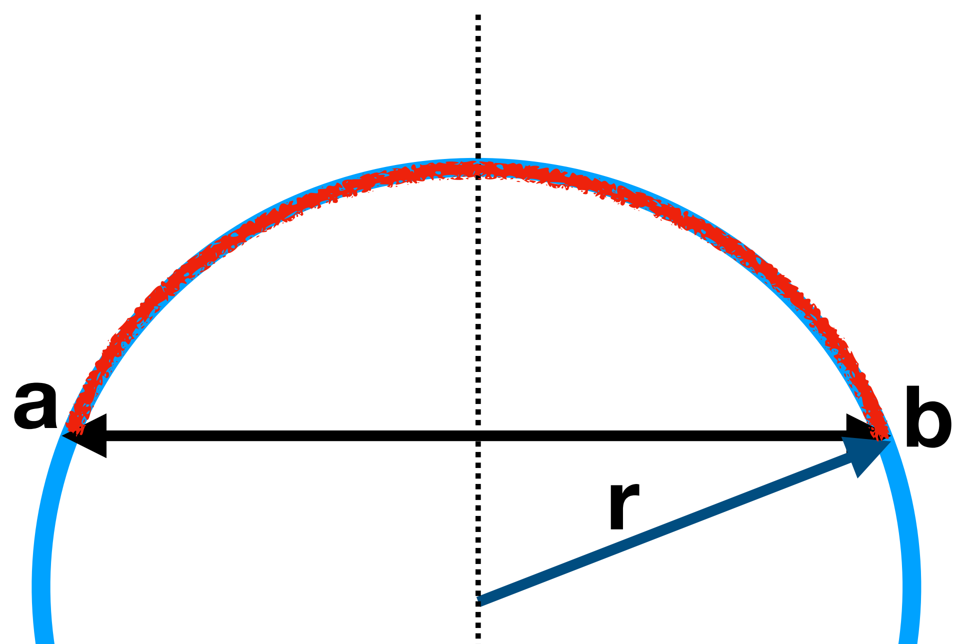
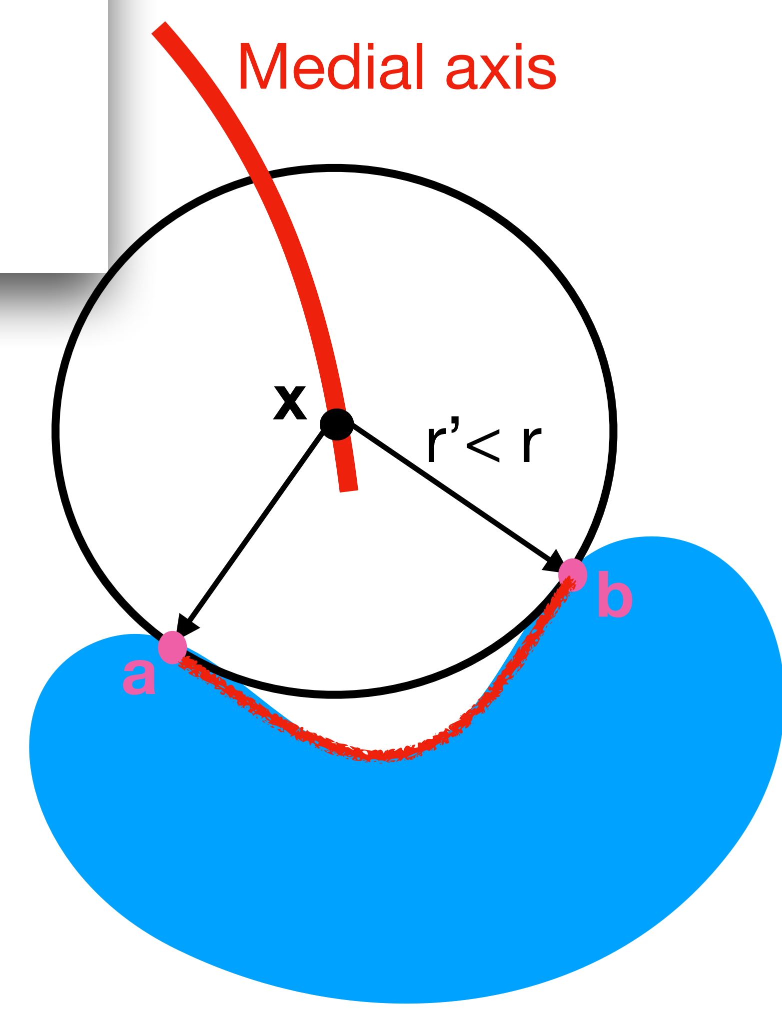
where the sup *over the empty set is* 0.

If $\operatorname{rch} \mathcal{S} < r$ then there is x in the medial axis with at least two points $a, b \in \mathcal{S}$ with $d(x, \mathcal{S}) = d(x,a) = d(x,b) = r' < r$ and:

$$|a-b| < 2r \quad \text{and} \quad d_{\mathcal{S}}(a,b) \geq 2r' \arcsin \frac{|a-b|}{2r'} > 2r \arcsin \frac{|a-b|}{2r}$$





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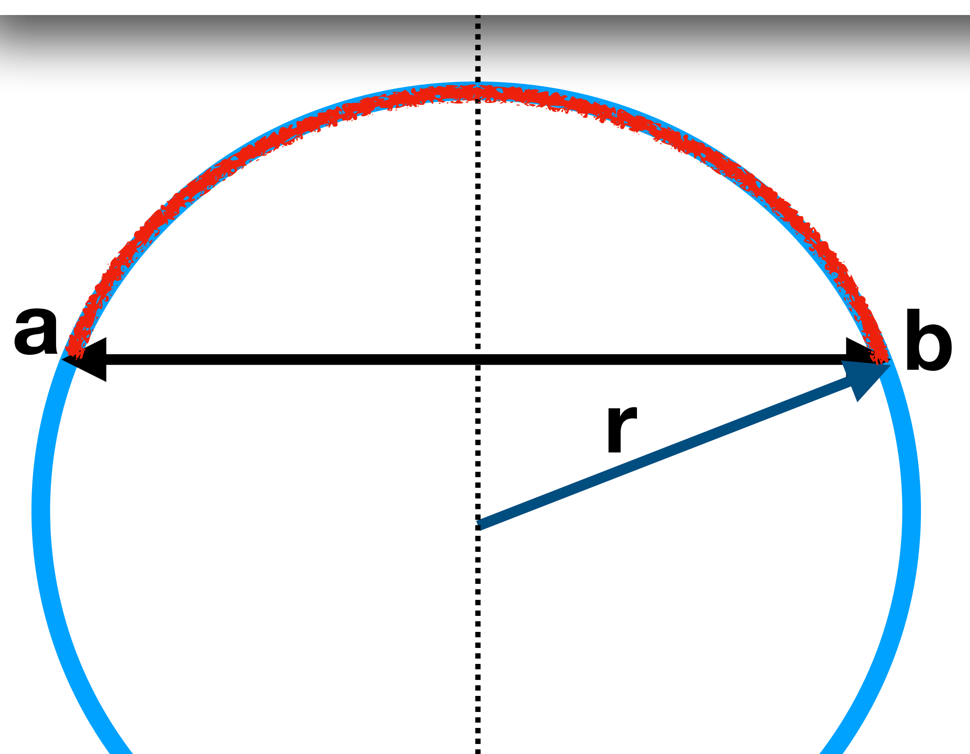

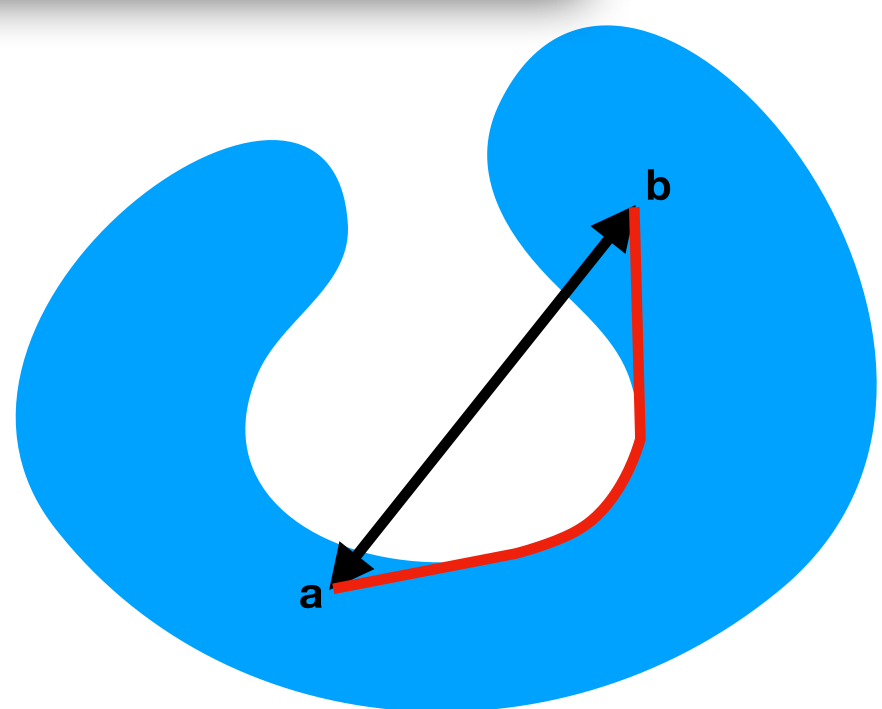

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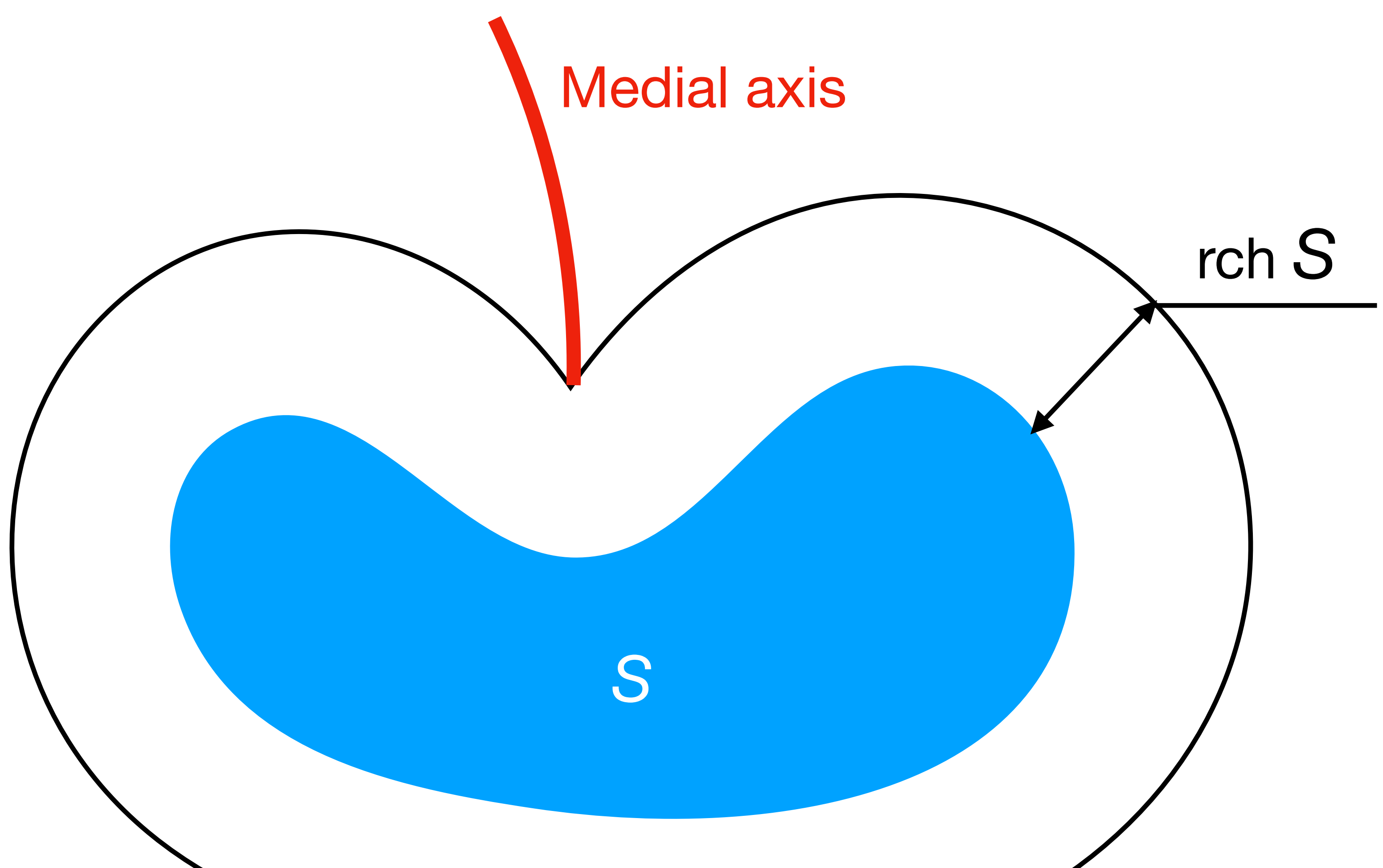

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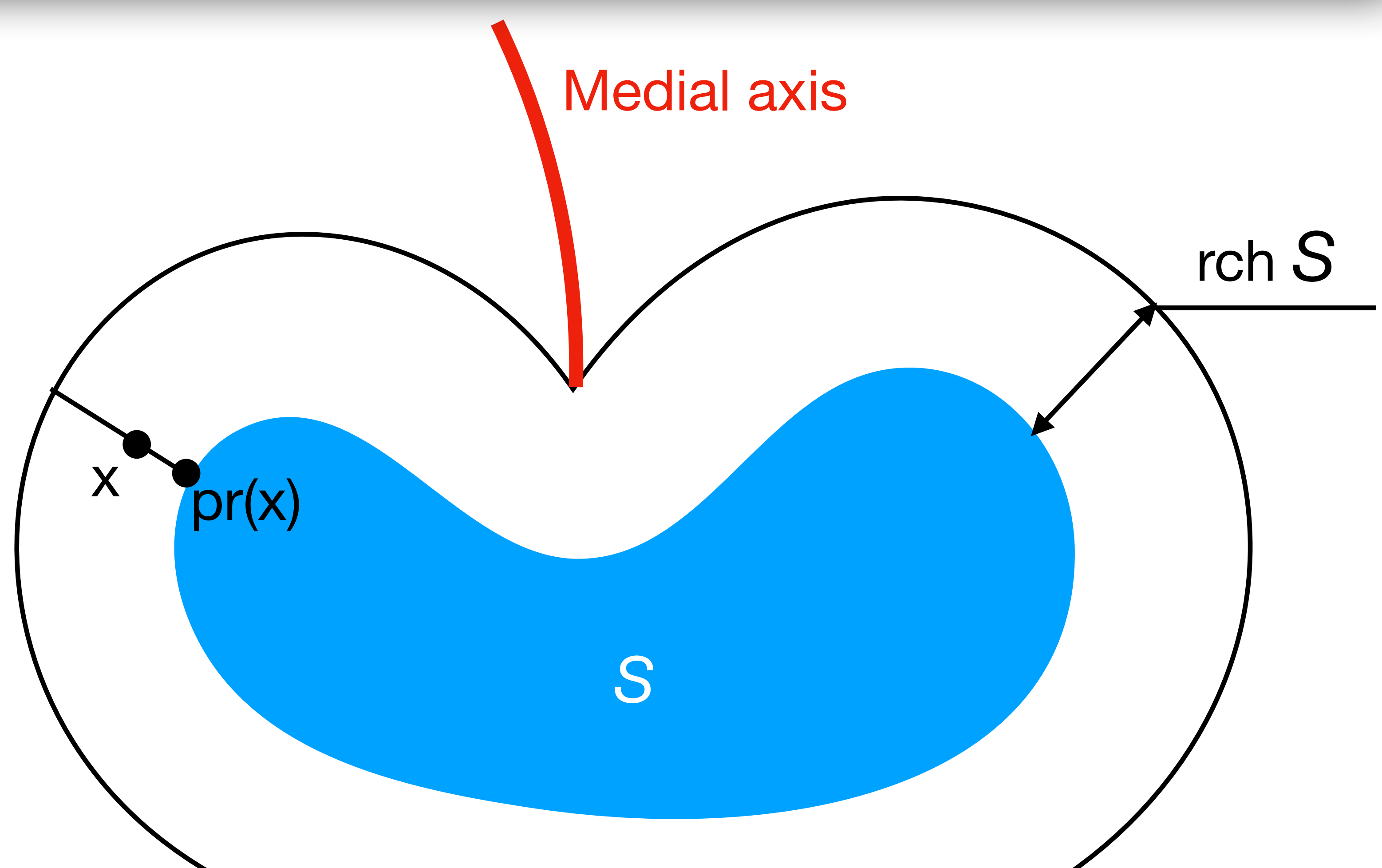
1) Tubular neighborhood:

If $0 < d(x, \mathcal{S}) < \operatorname{rch} \mathcal{S}$ and $\operatorname{pr}(x)$ is the point in $\mathcal{S}$ closest to x then:

$$\forall \lambda \in [0, \operatorname{rch} \mathcal{S}),\ \operatorname{pr}\left(\operatorname{pr}(x) + \lambda \frac{x - \operatorname{pr}(x)}{\|x - \operatorname{pr}(x)\|}\right) = \operatorname{pr}(x)$$



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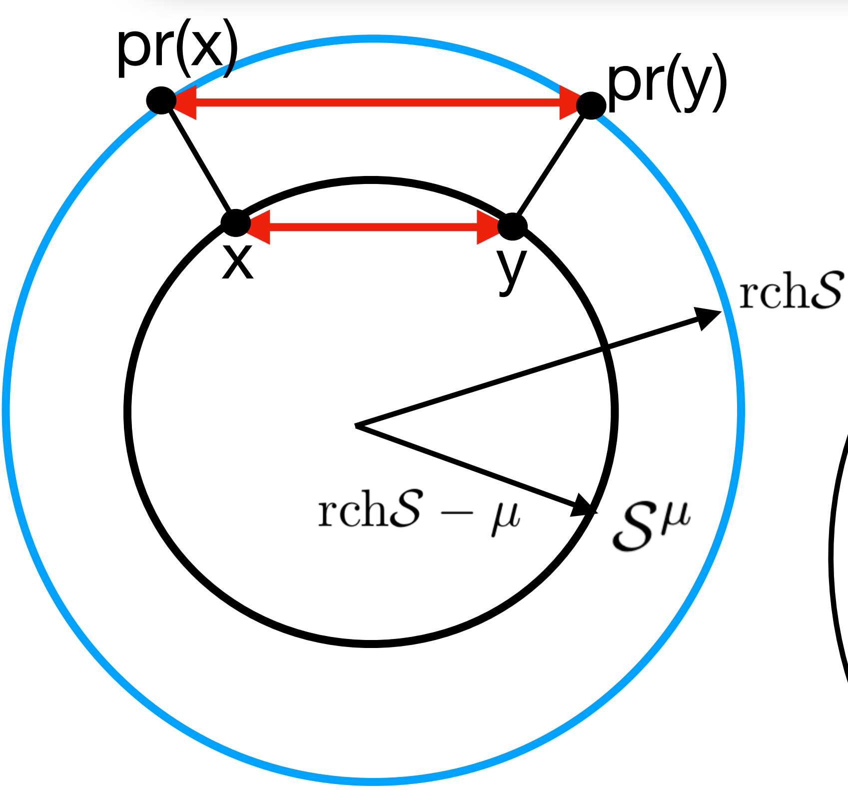
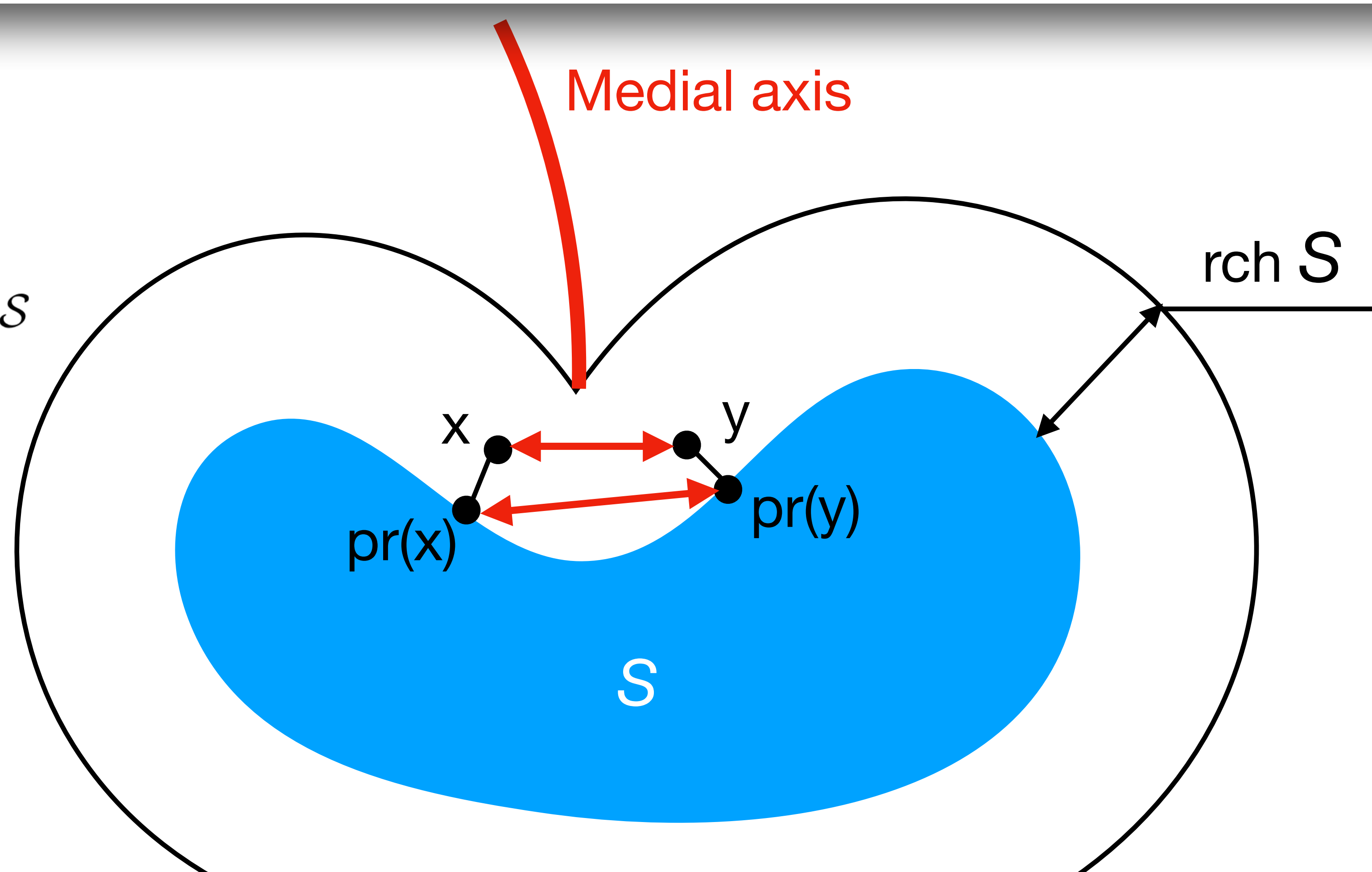
2) Projection is Lipschitz:

For $\mu < r = \mathrm{rch}\mathcal{S}$ the restriction of pr to the μ-tubular neighbourhood $\mathcal{S}^\mu$ is Lipschitz with constant:

$$\frac{\mathrm{rch}\mathcal{S}}{\mathrm{rch}\mathcal{S} - \mu}$$





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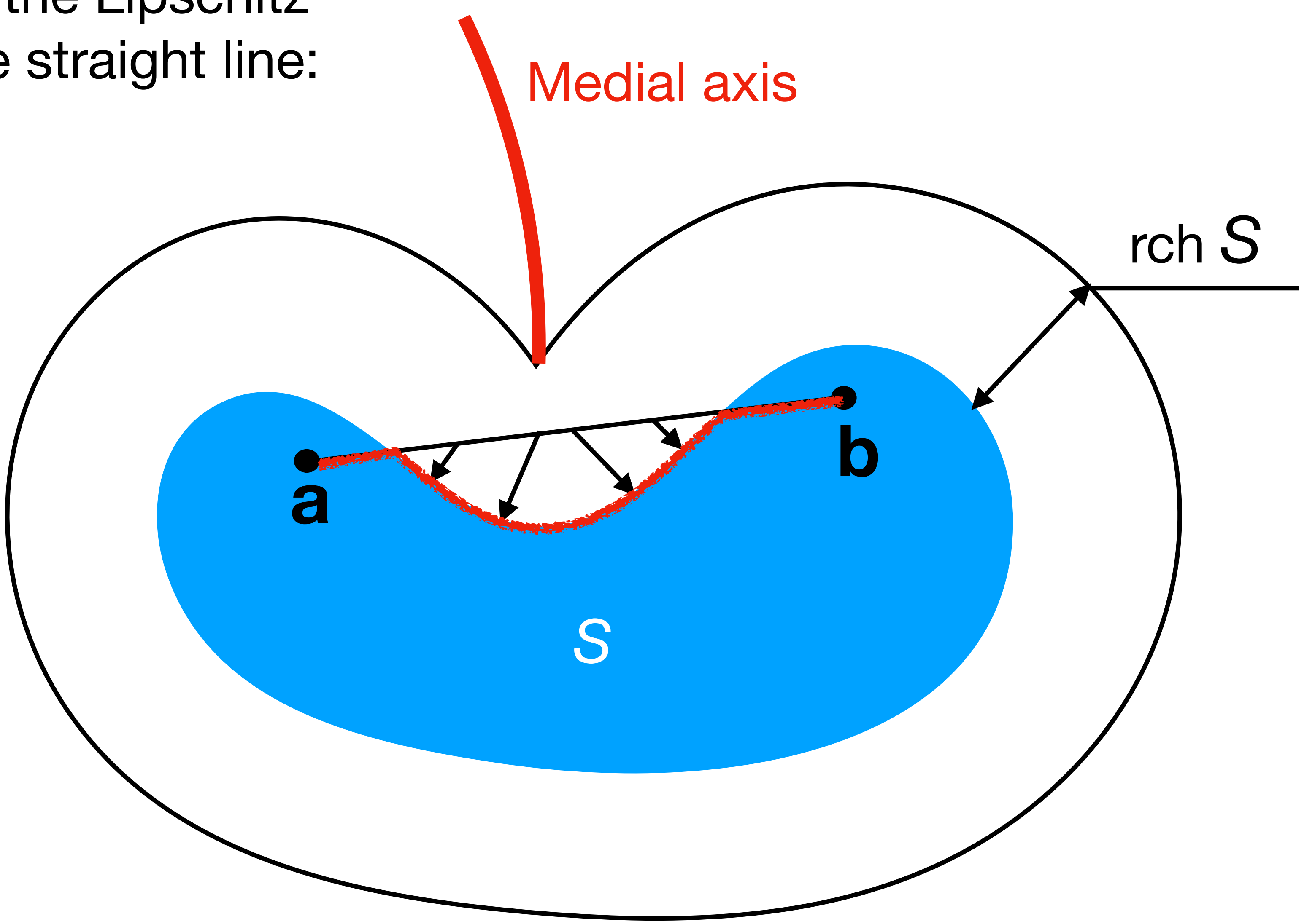

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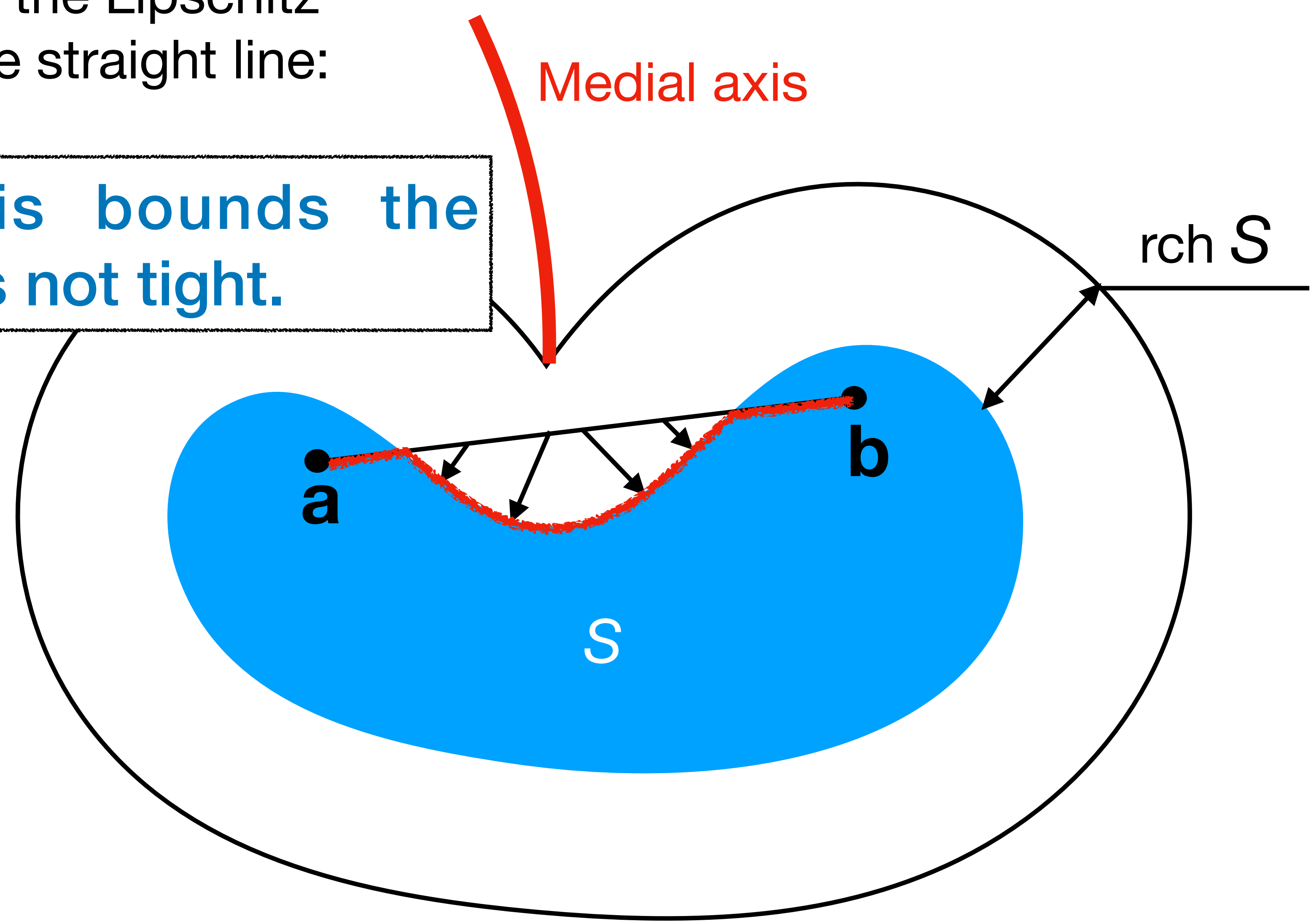
Unfortunately, even if this bounds the geodesic length, the bound is not tight.



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But this works :

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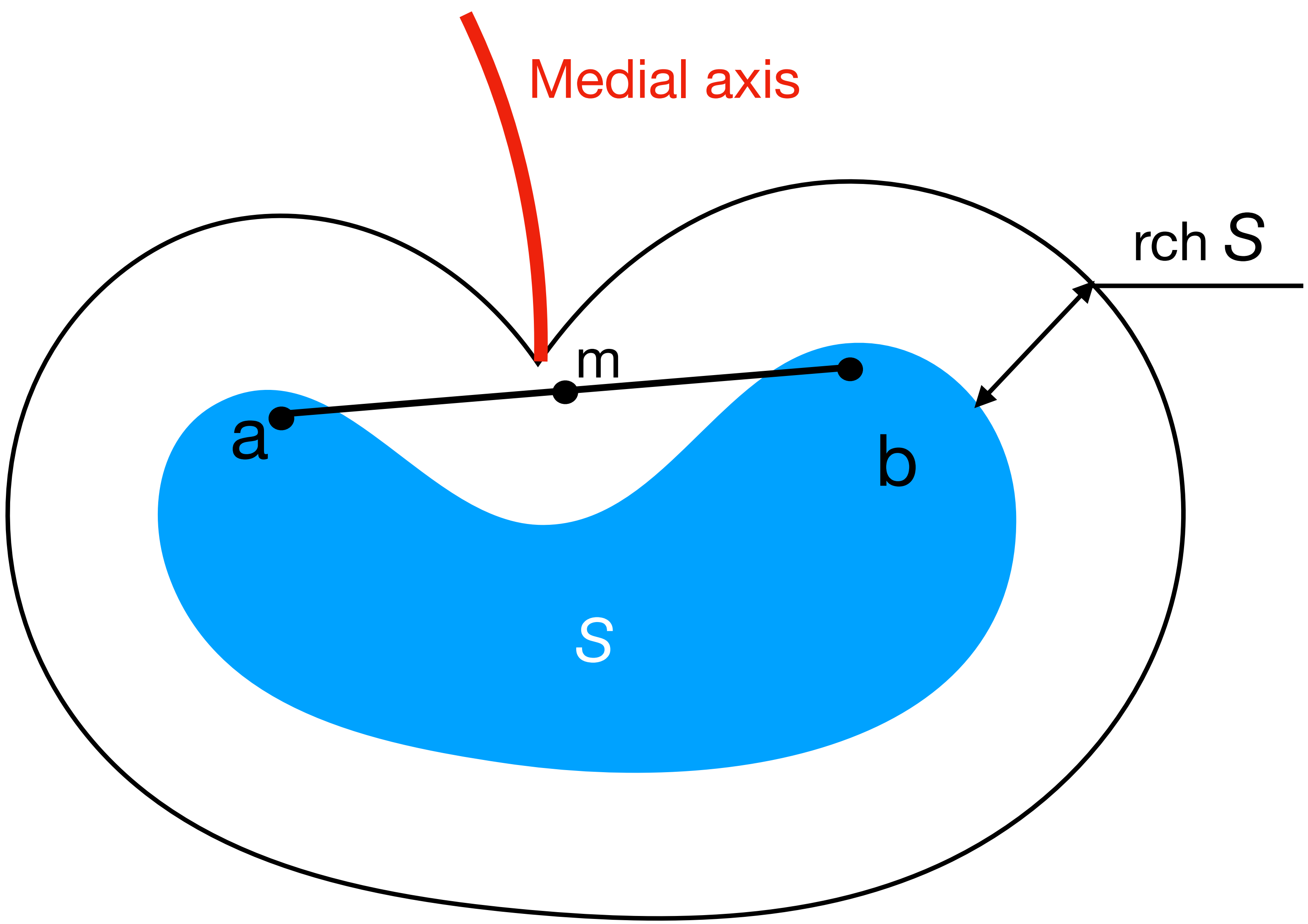

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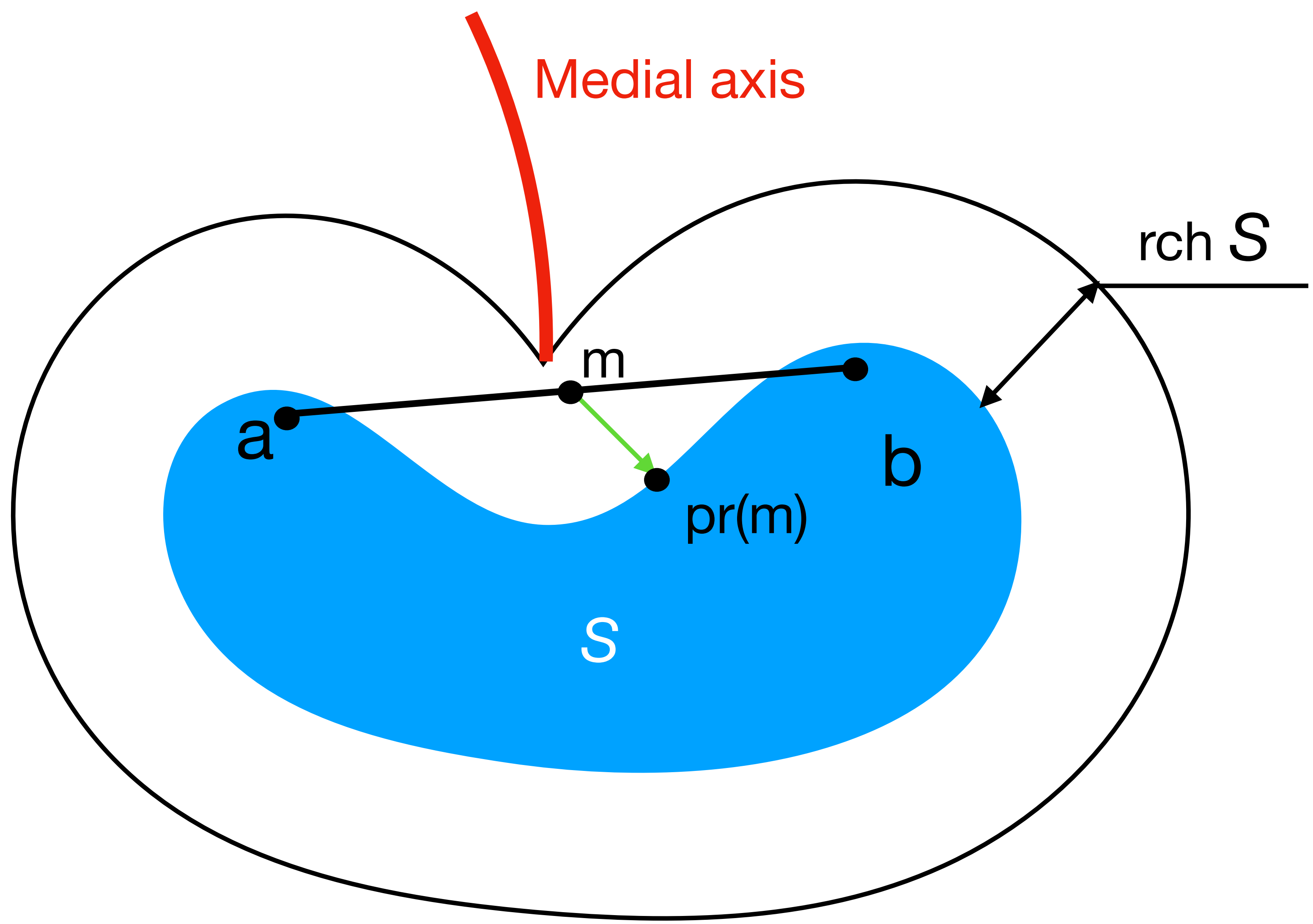

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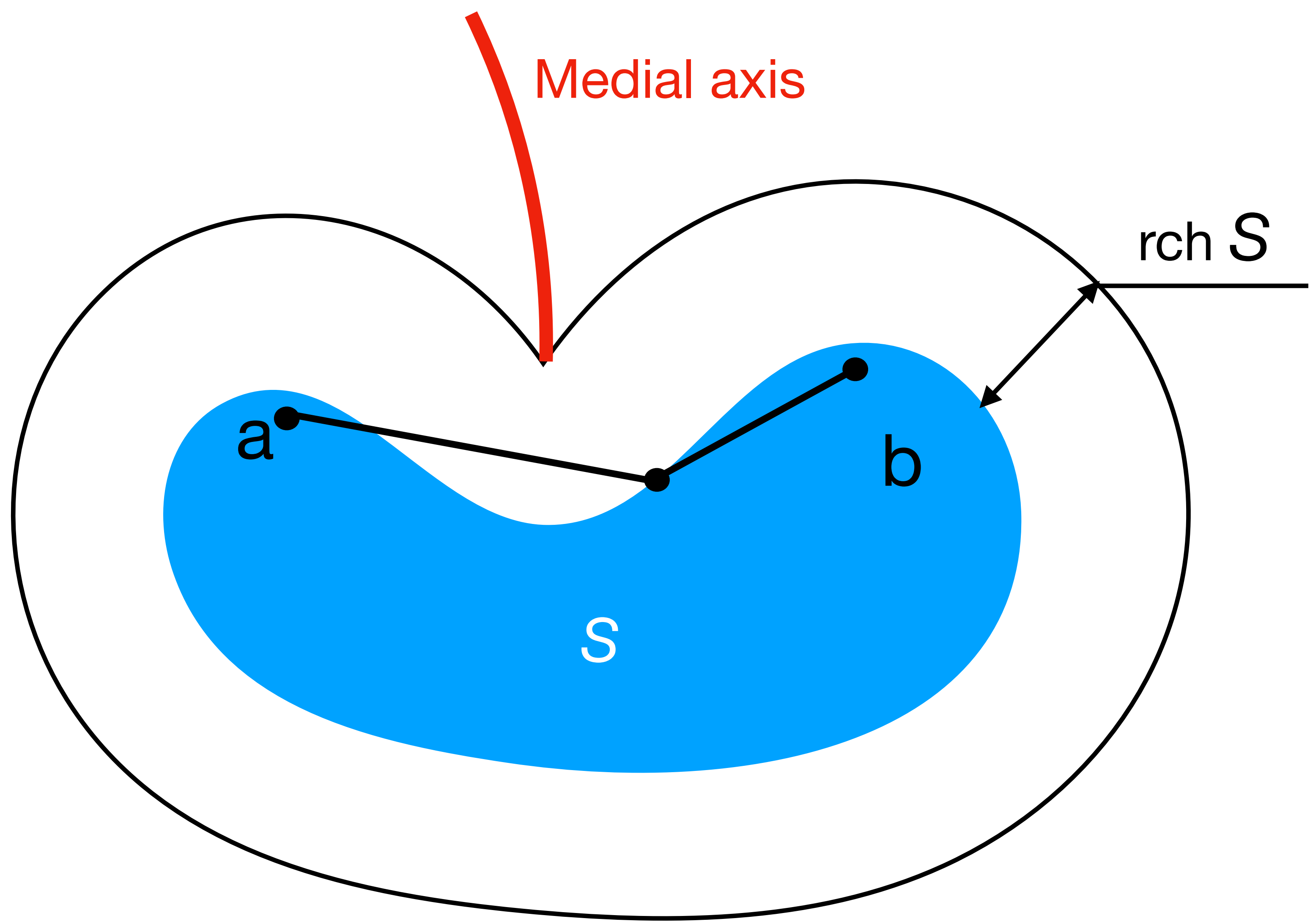

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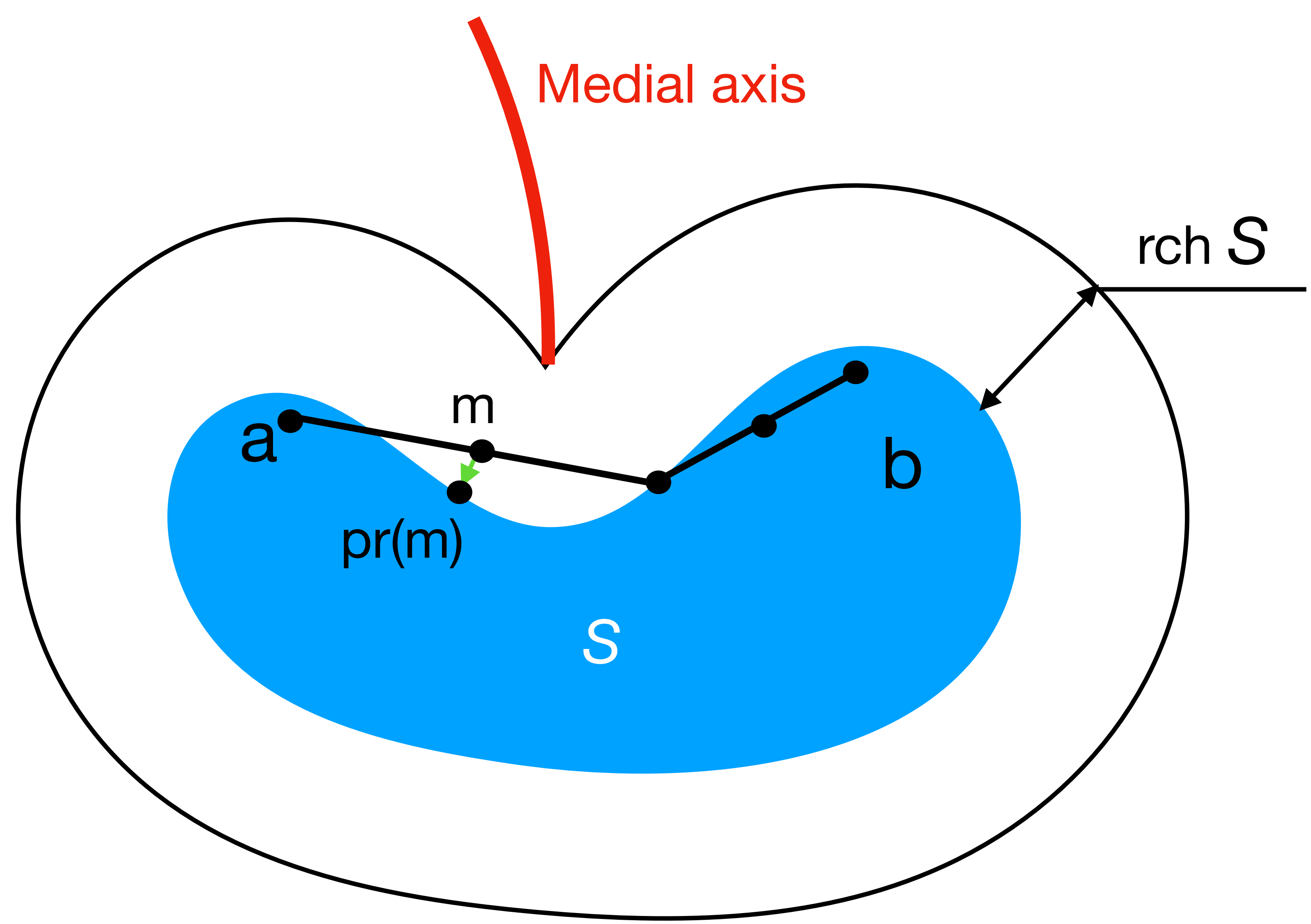

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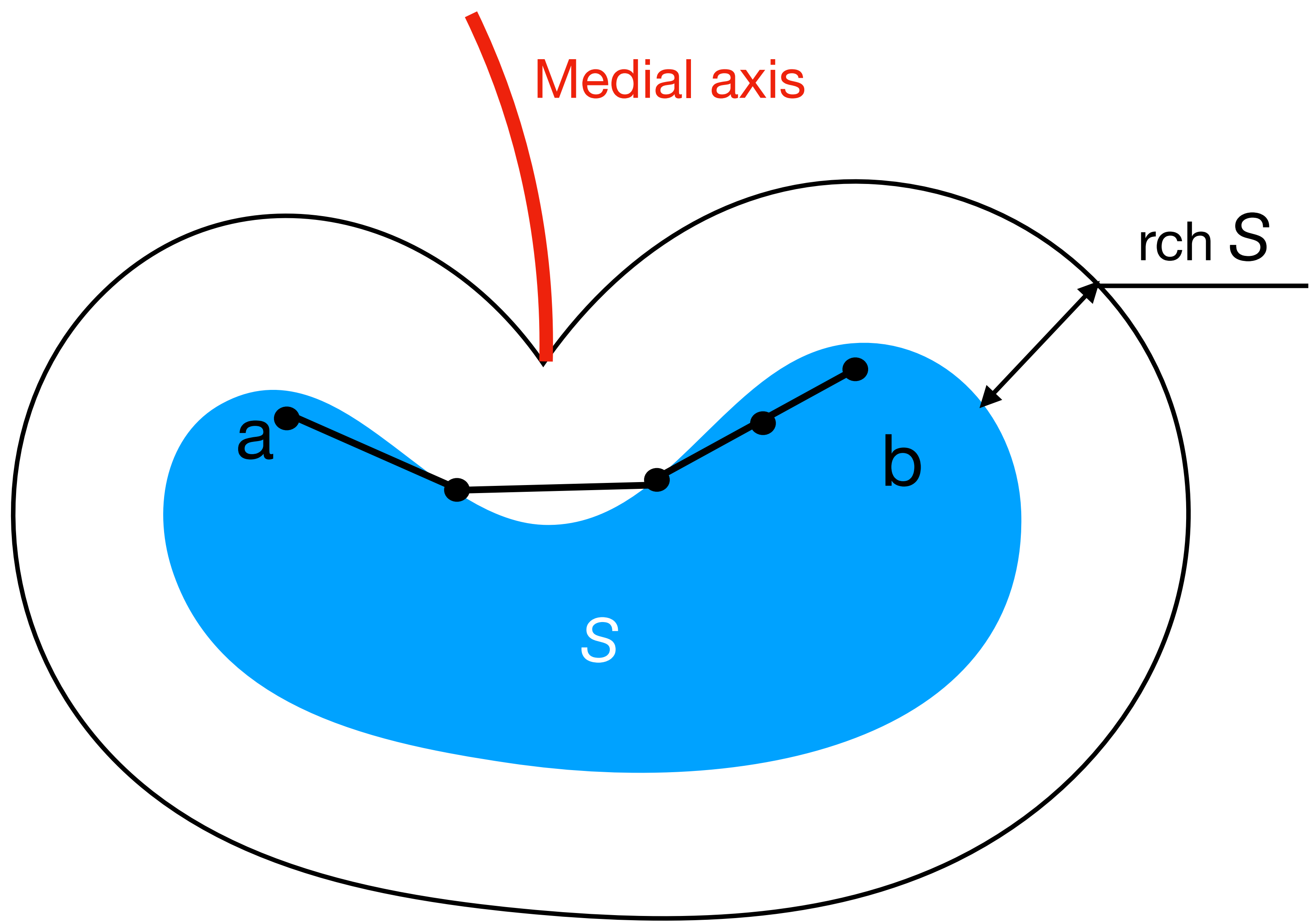

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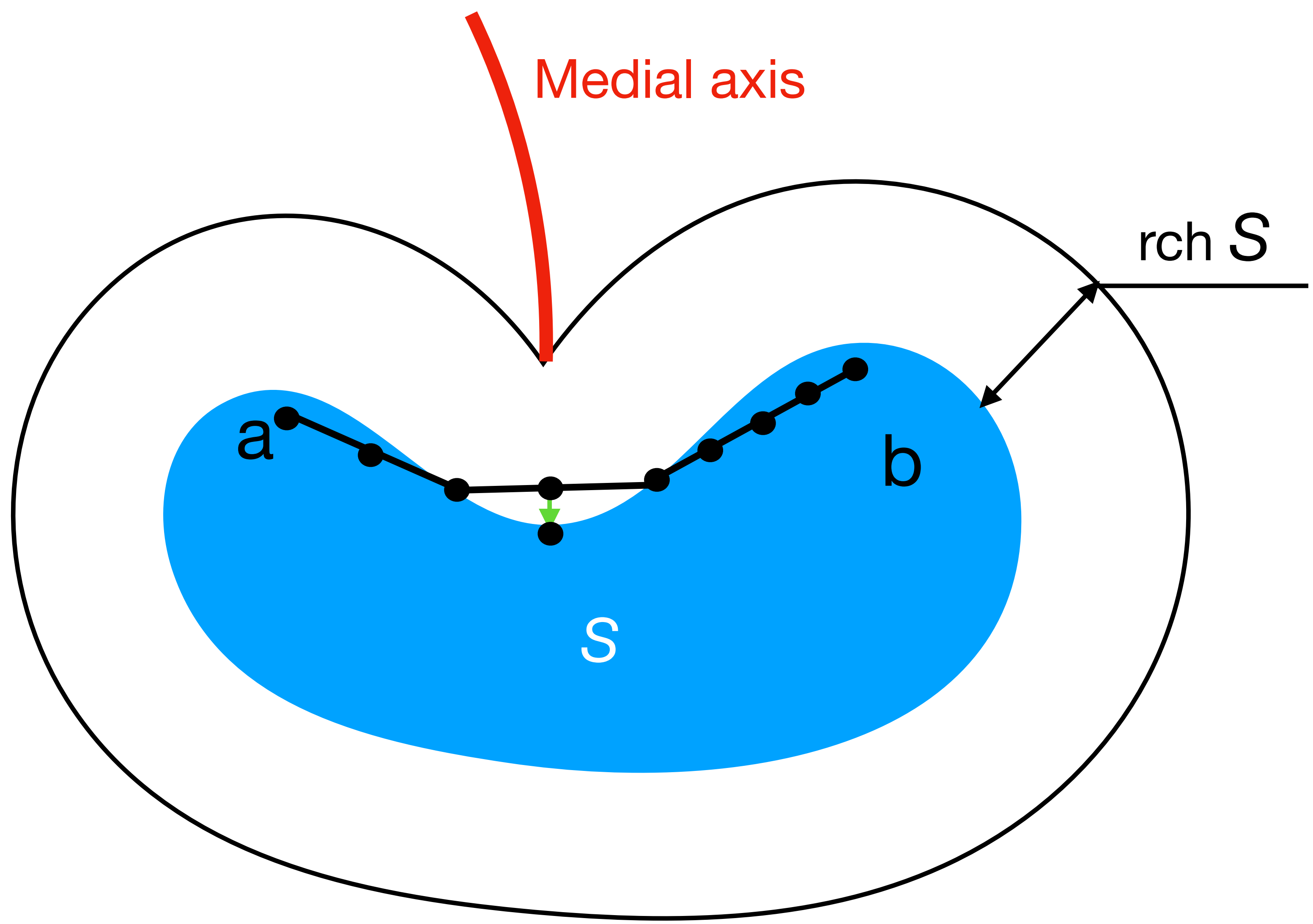

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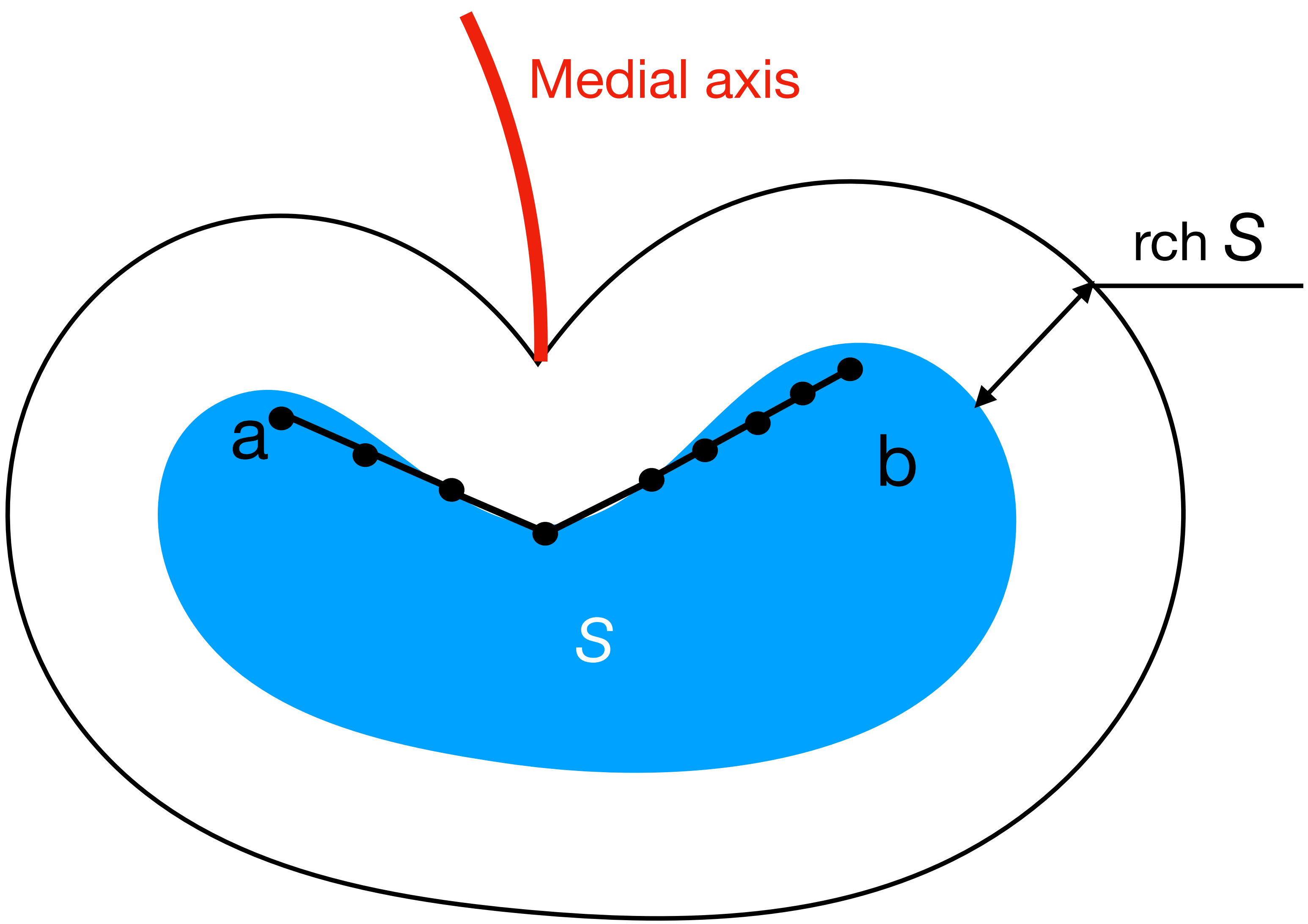

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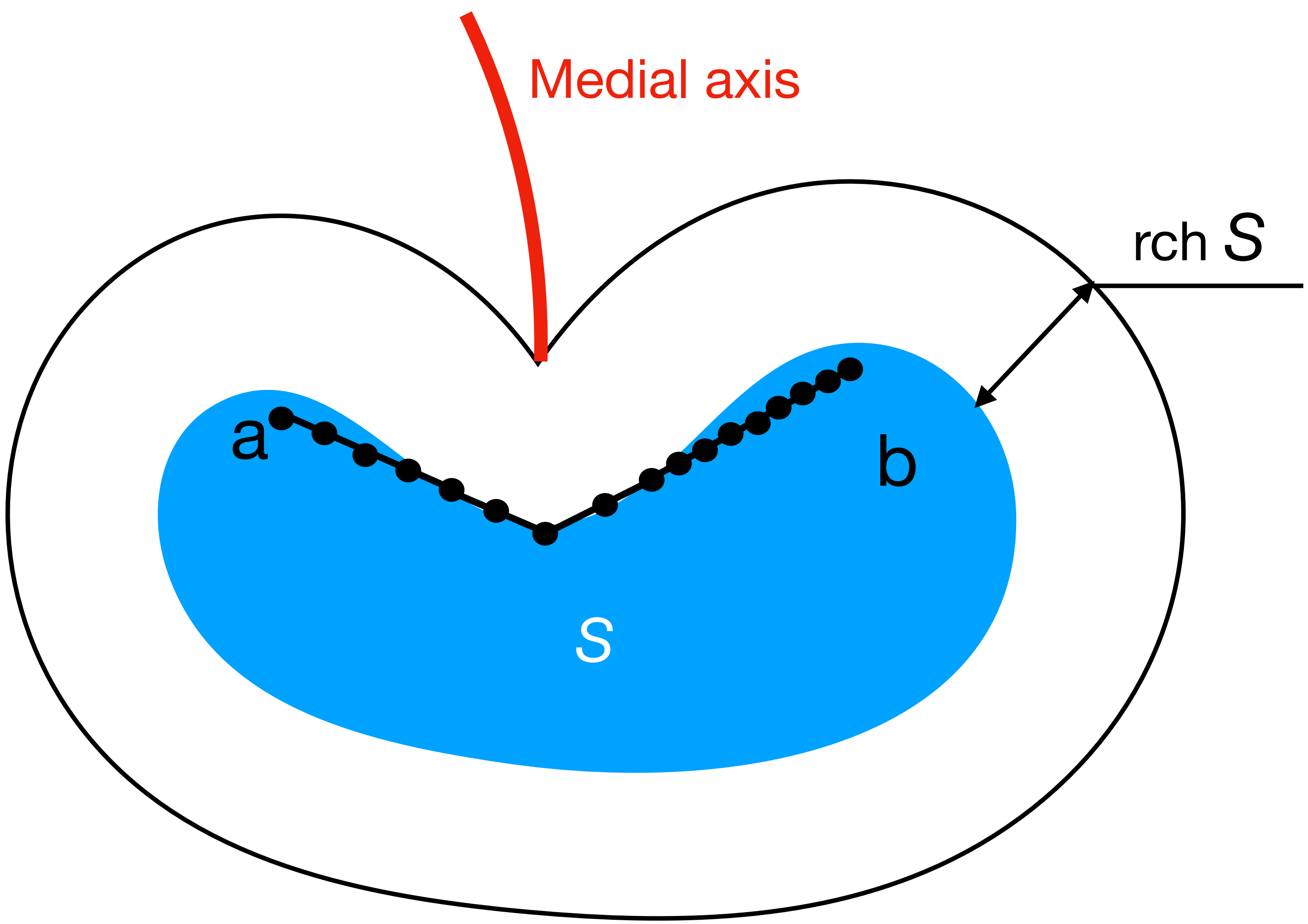
Lemma 6. *Let $\mathcal{S} \subset \mathbb{R}^d$ be a closed set with reach $r = \operatorname{rch} \mathcal{S} > 0$. For any $a, b \in \mathcal{S}$ such that $\|a - b\| < 2r$ one has:*

$$d_{\mathcal{S}}(a, b) \leq 2r \arcsin \frac{\|a - b\|}{2r}$$

But this works :

Step 4



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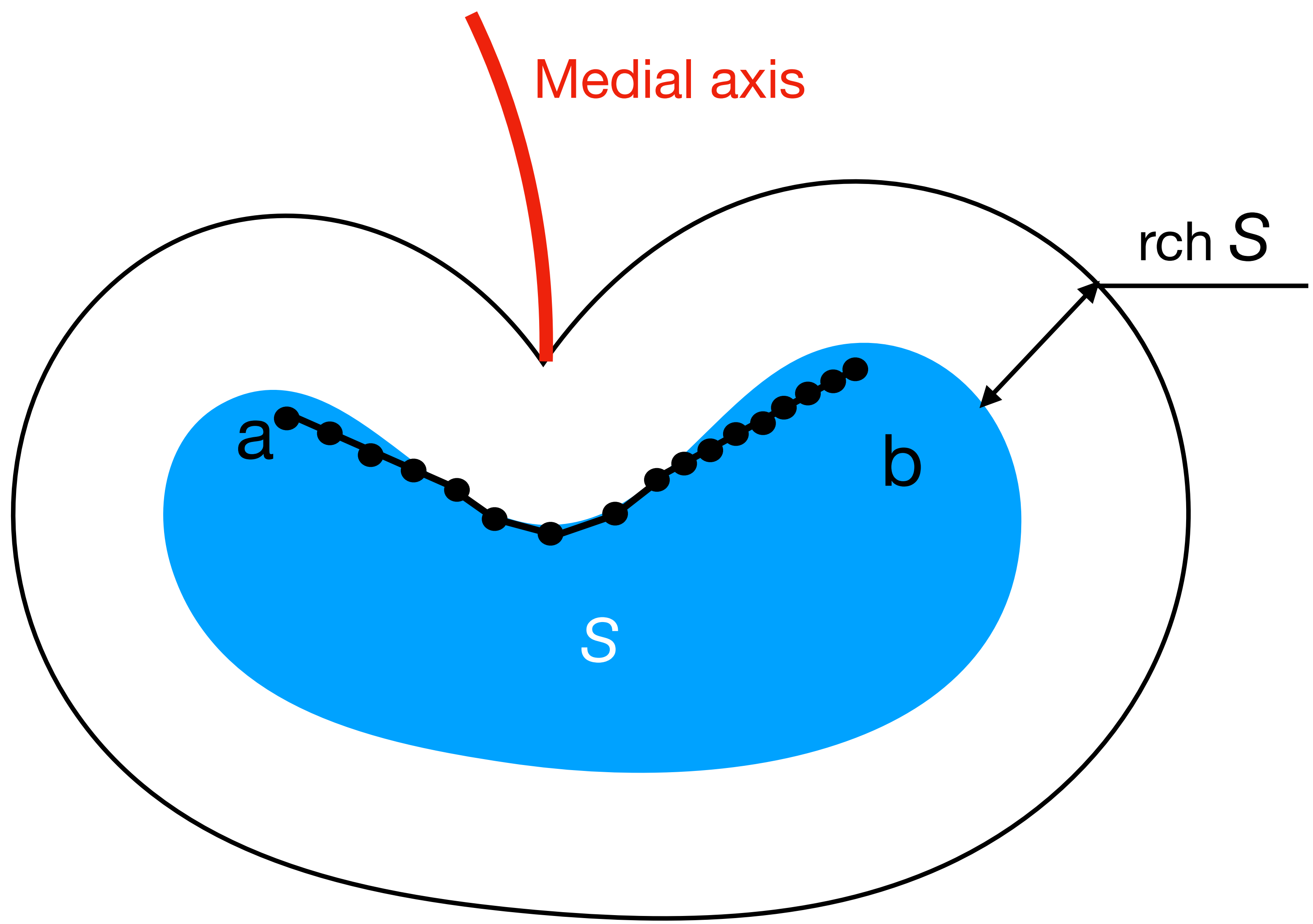

Proof of Theorem 1

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Lemma 5. *Let $\mathcal{S} \subset \mathbb{R}^d$ be a closed set with reach $r = \mathrm{rch}\,\mathcal{S} > 0$. For $a, b \in \mathcal{S}$ such that $\delta = \frac{\|a-b\|}{2} < r$ and $m = \frac{a+b}{2}$ one has:*

$$\|\pi_{\mathcal{S}}(m) - m\| \leq \rho$$

with:

$$\rho = r - \sqrt{r^2 - \delta^2}$$

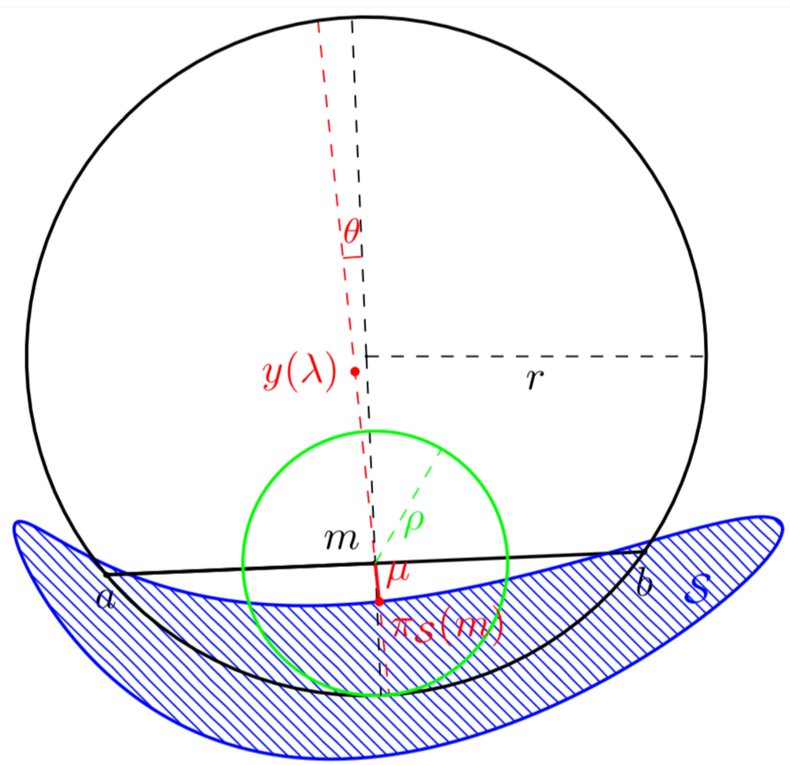
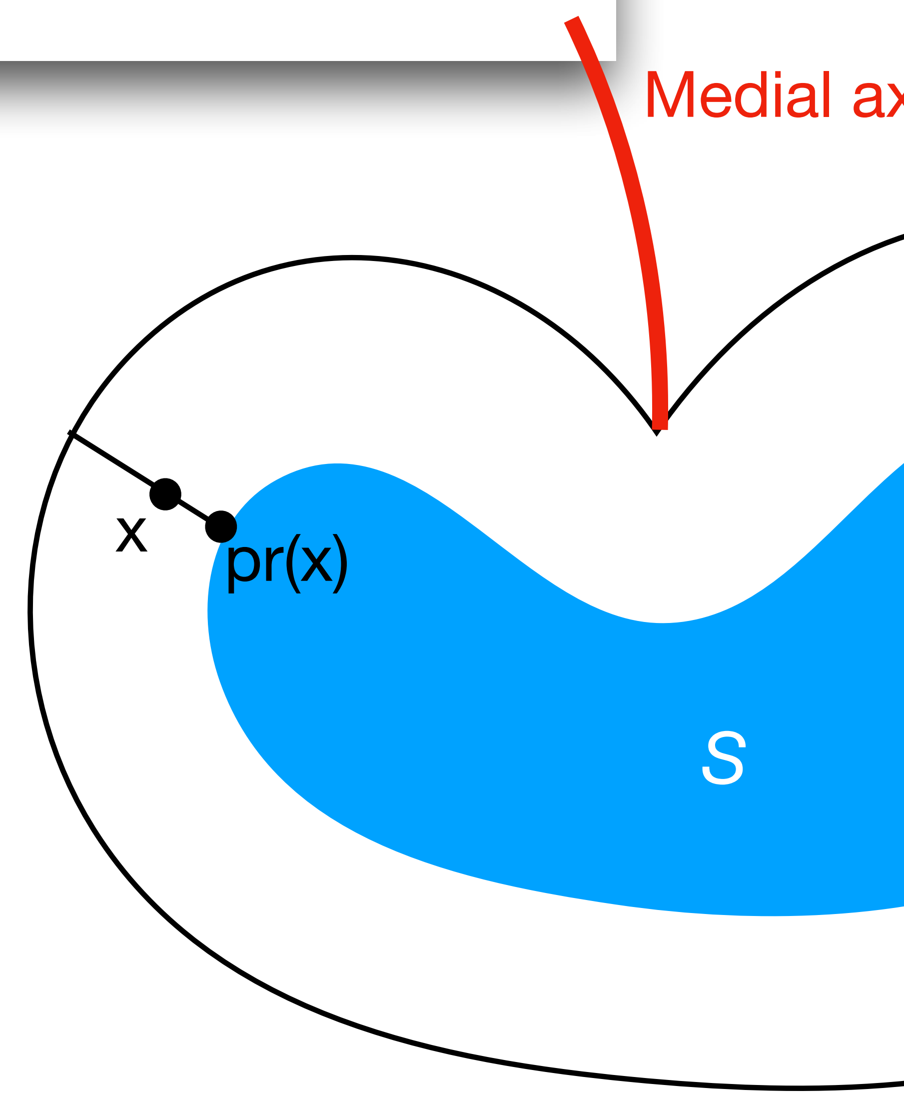


Figure 1 On the left the projection $\pi_{\mathcal{S}}(m)$ is contained in the disk of center m and radius ρ. The notation used in the proof of Lemma 3 is also added. From the right figure it is easy to deduce that $\rho = r - \sqrt{r^2 - \delta^2}$.



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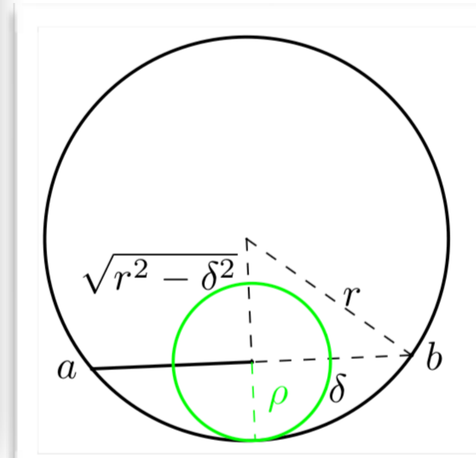
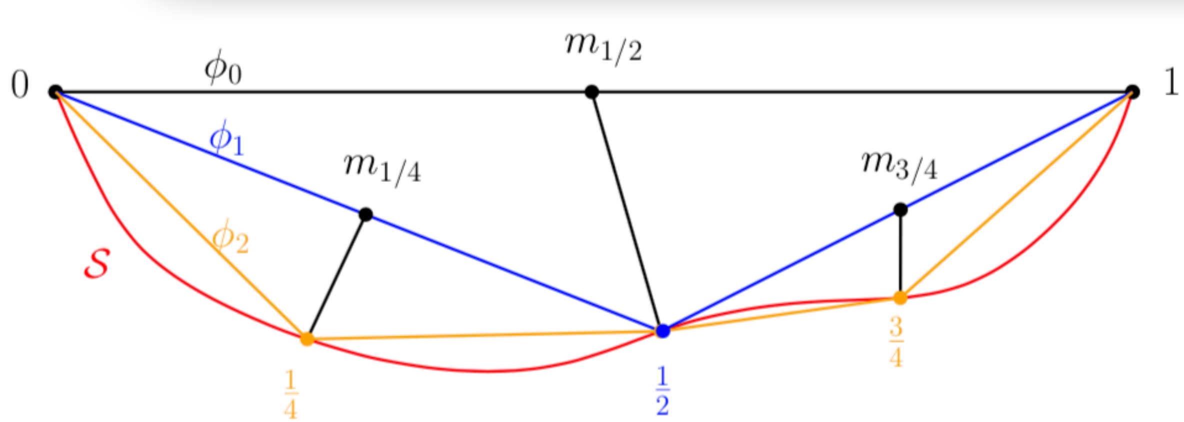
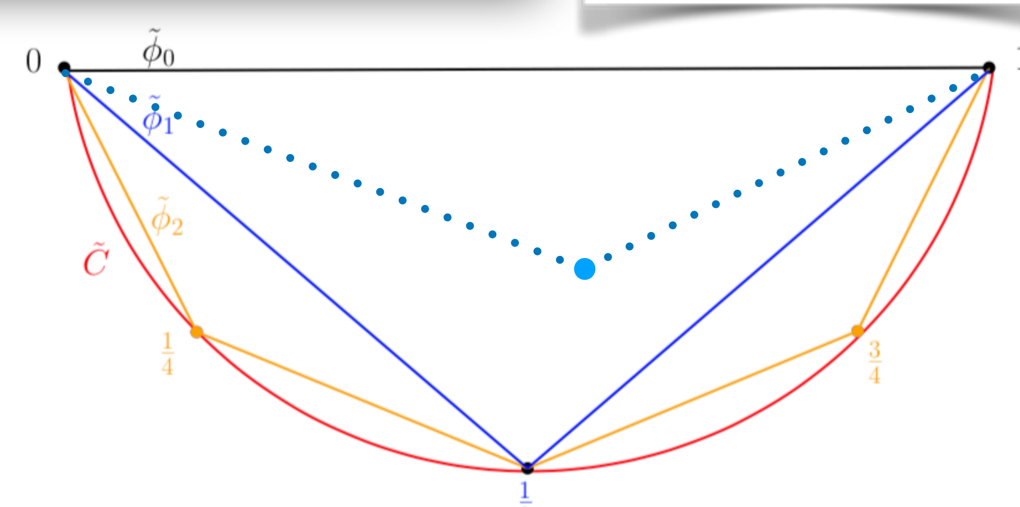
$$\begin{aligned} \operatorname{length}(\phi_i) &= \sum_{k=0}^{2^i - 1} |\phi_i((k+1)/2^i) - \phi_i(k/2^i)| \\ &\leq \sum_{k=0}^{2^i - 1} |\tilde{\phi}_i((k+1)/2^i) - \tilde{\phi}_i(k/2^i)| \\ &= \operatorname{length}(\tilde{\phi}_i) \leq \operatorname{length}(\tilde{C}_{\tilde{a},\tilde{b}}) = 2r \arcsin \frac{|a-b|}{2r}. \end{aligned}$$







Proof of Theorem 1

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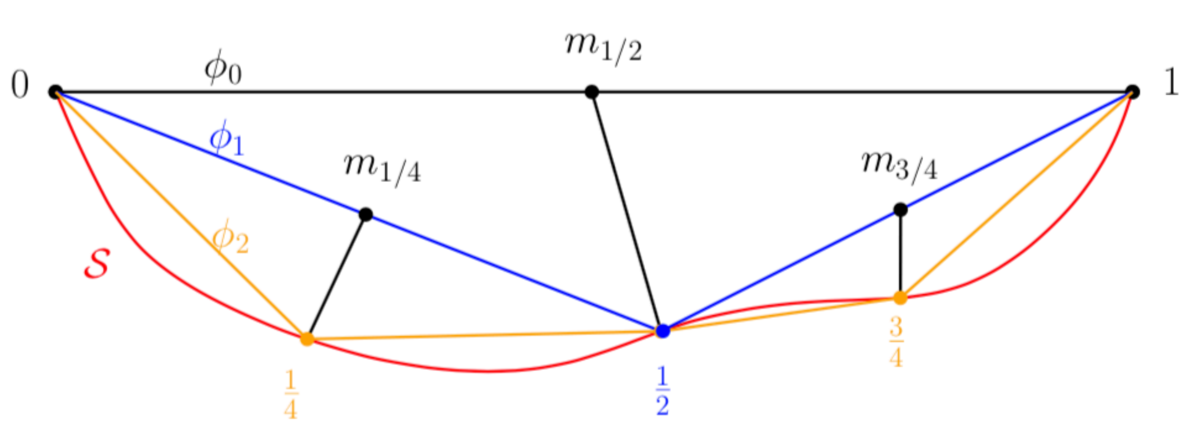
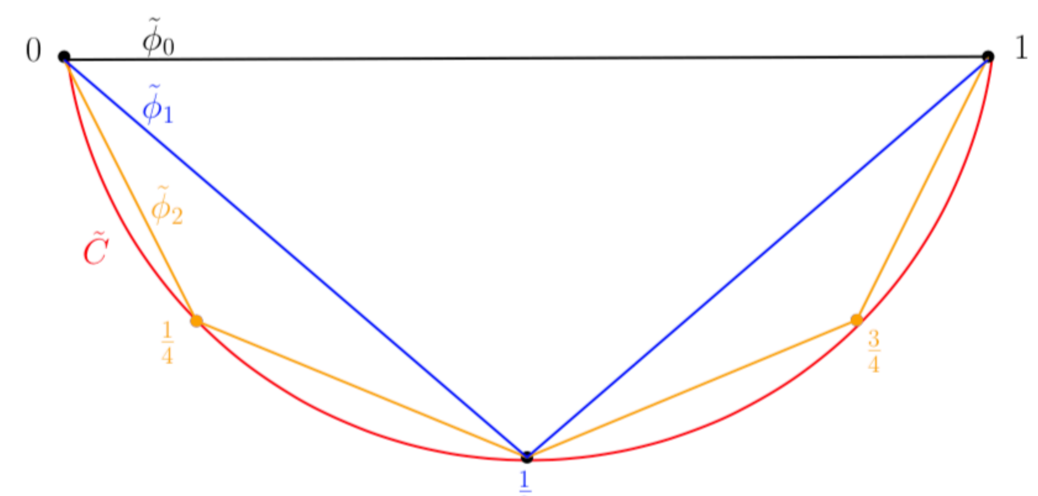
$$\delta_i = \frac{1}{2} \max_{0 \leq k \leq 2^i - 1} |\phi_i((k+1)/2^i) - \phi_i(k/2^i)|.$$

$$\lim_{i \to \infty} \delta_i = 0.$$

$$\operatorname{length}(\pi_{\mathcal{S}} \circ \phi_i) \leq \frac{\operatorname{rch}\mathcal{S}}{\operatorname{rch}\mathcal{S} - \delta_i} \operatorname{length}(\phi_i) \leq \frac{\operatorname{rch}\mathcal{S}}{\operatorname{rch}\mathcal{S} - \delta_i} 2r \arcsin \frac{|a-b|}{2r}$$

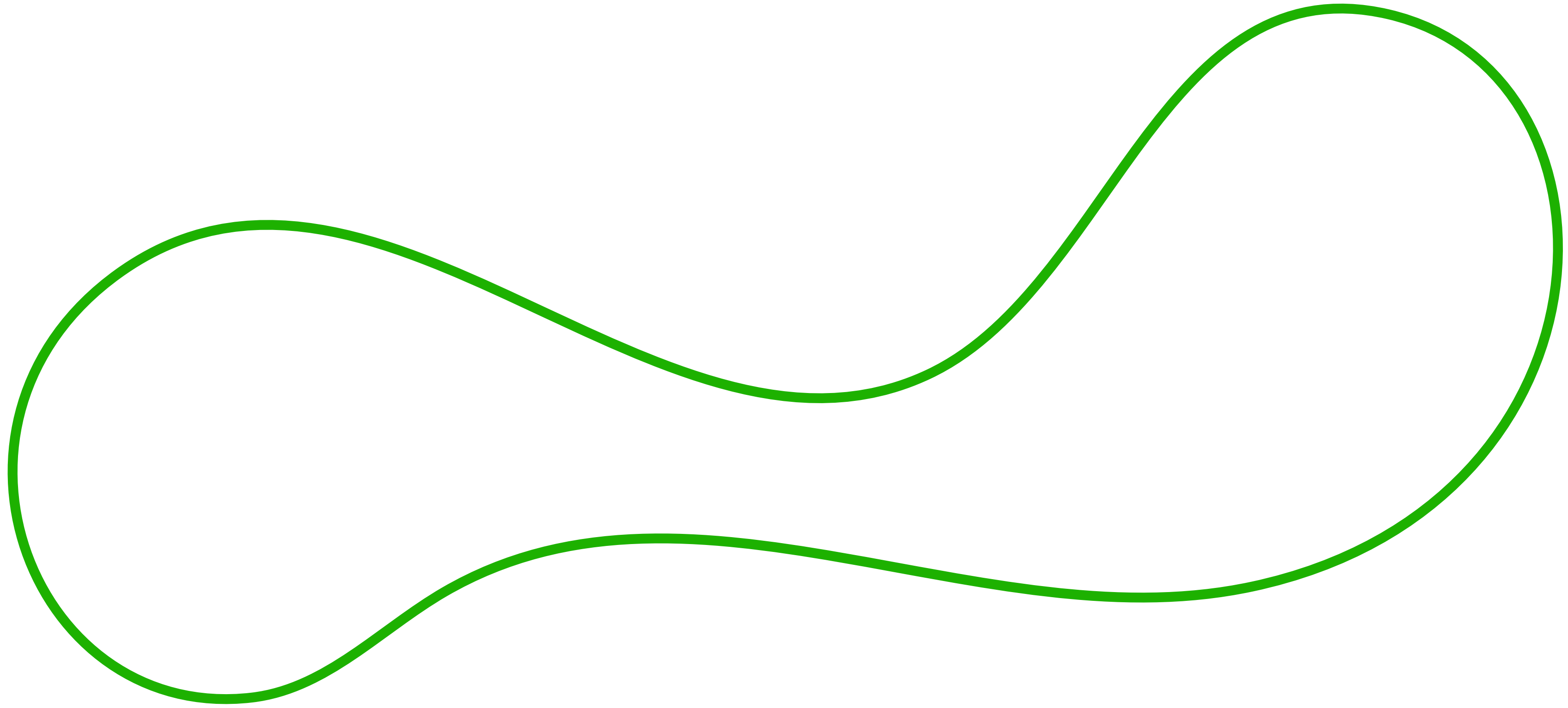




Embedded manifolds with positive reach

If $\mathcal{M}$ is a $C^{1,1}$ compact manifold embedded in R^d then $\operatorname{rch}\mathcal{M} > 0$

If $\mathcal{M}$ is a manifold embedded in R^d with $\operatorname{rch}\mathcal{M} > 0$ then $\mathcal{M}$ is $C^{1,1}$

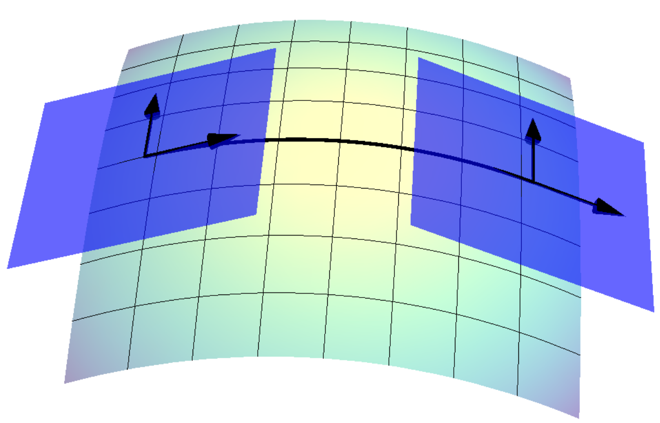


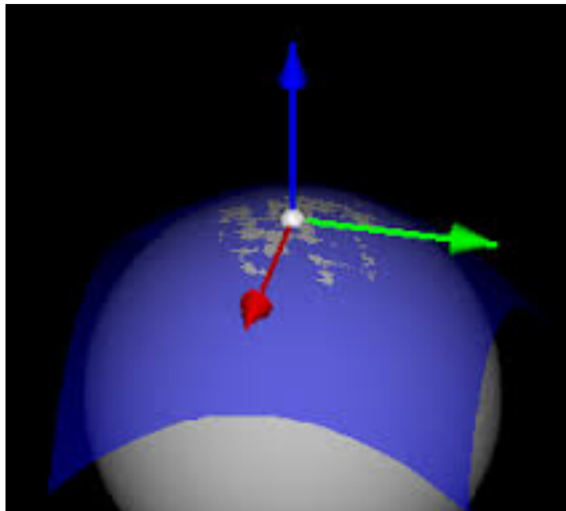
Reach and curvature

If $\mathcal{M}$ is a C^2 manifold embedded in R^d, and II_p denotes its second fundamental form at point $p \in \mathcal{M}$, then:

$$\|\mathrm{II}_p\| = \sup_{\|u\|=\|v\|=1} \|\mathrm{II}_p(u,v)\| \leq \sup_{\|w\|=1} \|\mathrm{II}_p(w,w)\| \leq \frac{1}{\operatorname{rch} \mathcal{M}}$$

▶ **Lemma 9.** *Let $\gamma(t)$ be a geodesic parametrized according to arc length on $\mathcal{M} \subset \mathbb{R}^d$, then $|\ddot{\gamma}| \leq 1/\operatorname{rch}(\mathcal{M})$, where we use Newton's notation, that is we write $\ddot{\gamma}$ for the second derivative of γ with respect to t.*





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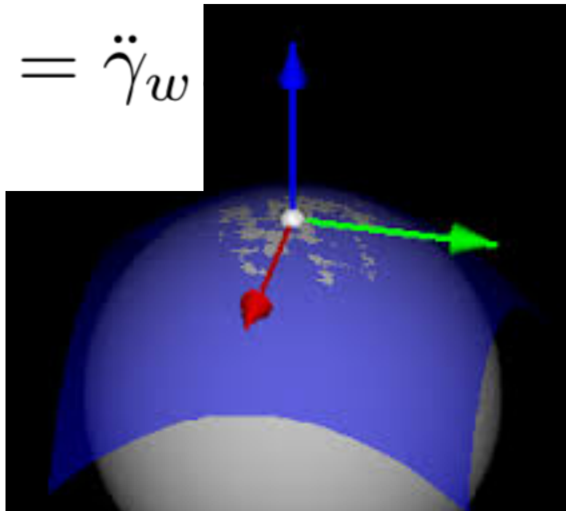
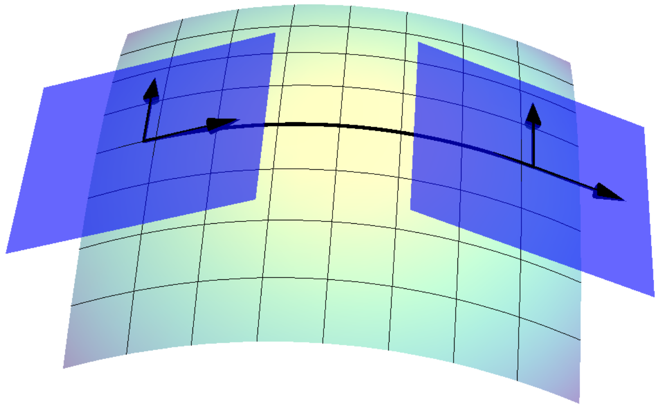
$$\mathrm{II}_p(w,w) = \mathrm{II}_p(\dot{\gamma}_w, \dot{\gamma}_w) = \bar{\nabla}_{\dot{\gamma}_w} \dot{\gamma}_w - \nabla_{\dot{\gamma}_w} \dot{\gamma}_w = \bar{\nabla}_{\dot{\gamma}_w} \dot{\gamma}_w - 0 = \ddot{\gamma}_w$$



Tangent variation on Manifolds



If $\mathcal{M}$ is a C^2 manifold embedded in R^d, then:

▶ **Lemma 11.** *Let* $p, q \in \mathcal{M}$, *then*

$$\angle(T_p\mathcal{M}, T_q\mathcal{M}) \leq \frac{d_{\mathcal{M}}(p,q)}{\mathrm{rch}(\mathcal{M})}.$$

Lemma 6.

$$d_{\mathcal{S}}(a,b) \leq 2r \arcsin \frac{\|a-b\|}{2r}$$

$$\sin\left(\frac{\angle(T_p\mathcal{M}, T_q\mathcal{M})}{2}\right) \leq \frac{|p-q|}{2\mathrm{rch}(\mathcal{M})}.$$

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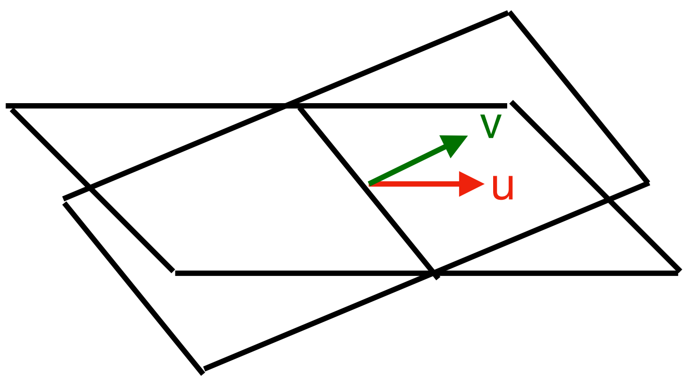


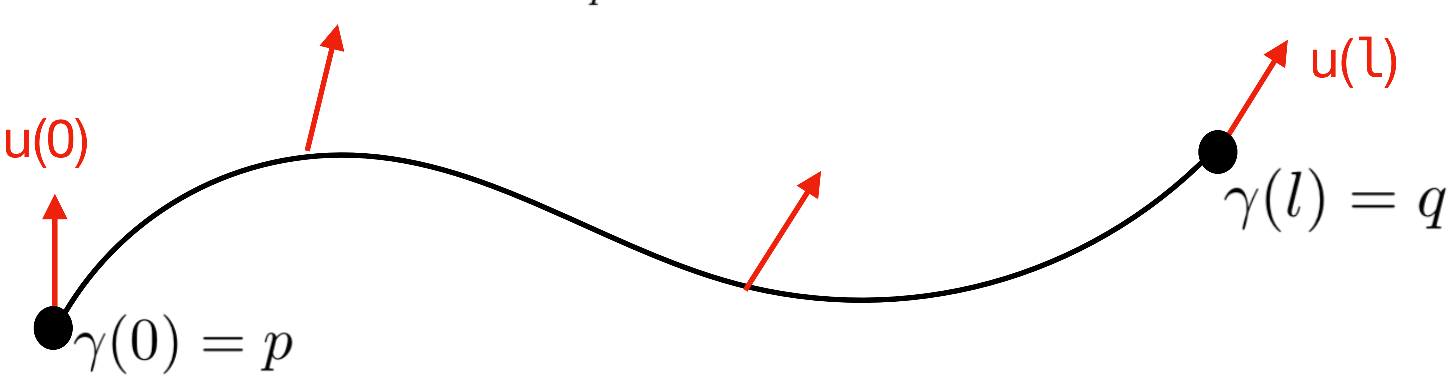
By definition:

$$\angle(T_p\mathcal{M}, T_q\mathcal{M}) = \sup_{u \in T_p\mathcal{M}} \inf_{v \in T_q\mathcal{M}} \angle u, v$$

And therefore if $d_{\mathcal{M}}(p,q) = l$ and γ is a geodesic parametrized by arc length such that $\gamma(0) = p$ and $\gamma(l) = q$,

if $t \mapsto u(t)$ is the parallel transport of a unit vector $u = u(0)$ along γ then:

$$\angle(T_p\mathcal{M}, T_q\mathcal{M}) \leq \sup_{u \in T_p\mathcal{M}} \angle u(0), u(l) \leq \int_0^l \frac{du}{dt} dt$$



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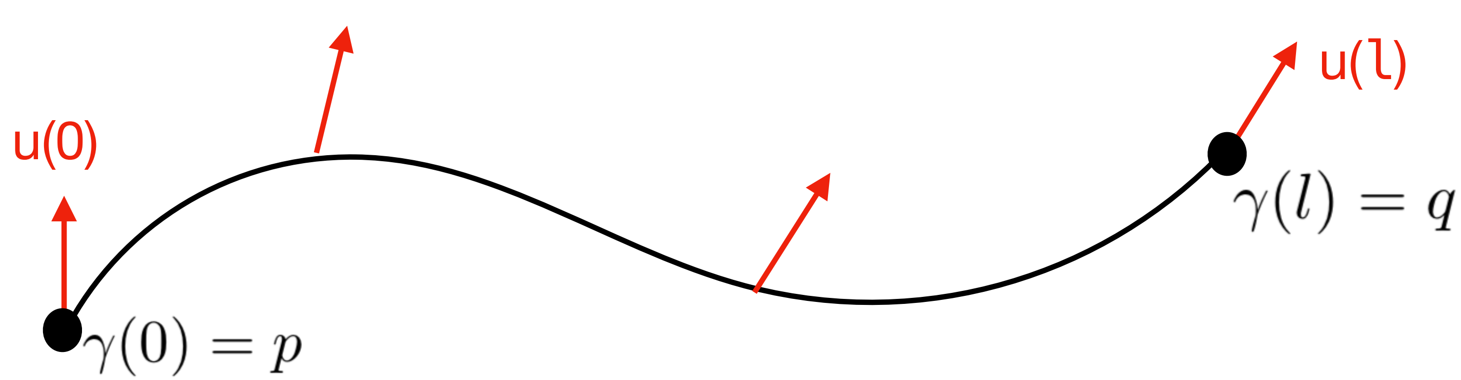
$$\frac{du}{dt} = \bar{\nabla}_{\dot{\gamma}} u(t) = \mathrm{II}_{\gamma(t)}(\dot{\gamma}, u(t)) + \nabla_{\dot{\gamma}} u(t) = \mathrm{II}_p(\dot{\gamma}, u(t))$$



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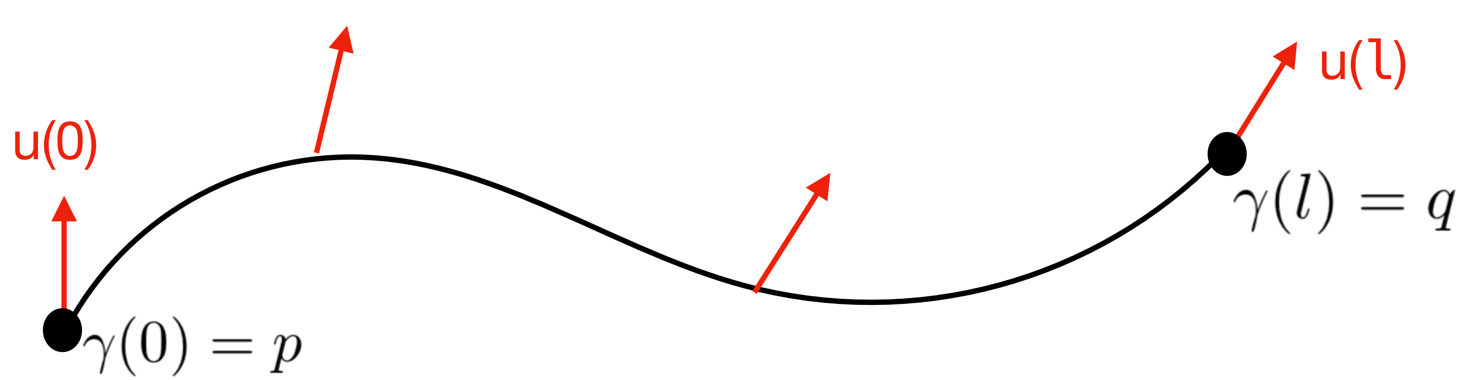
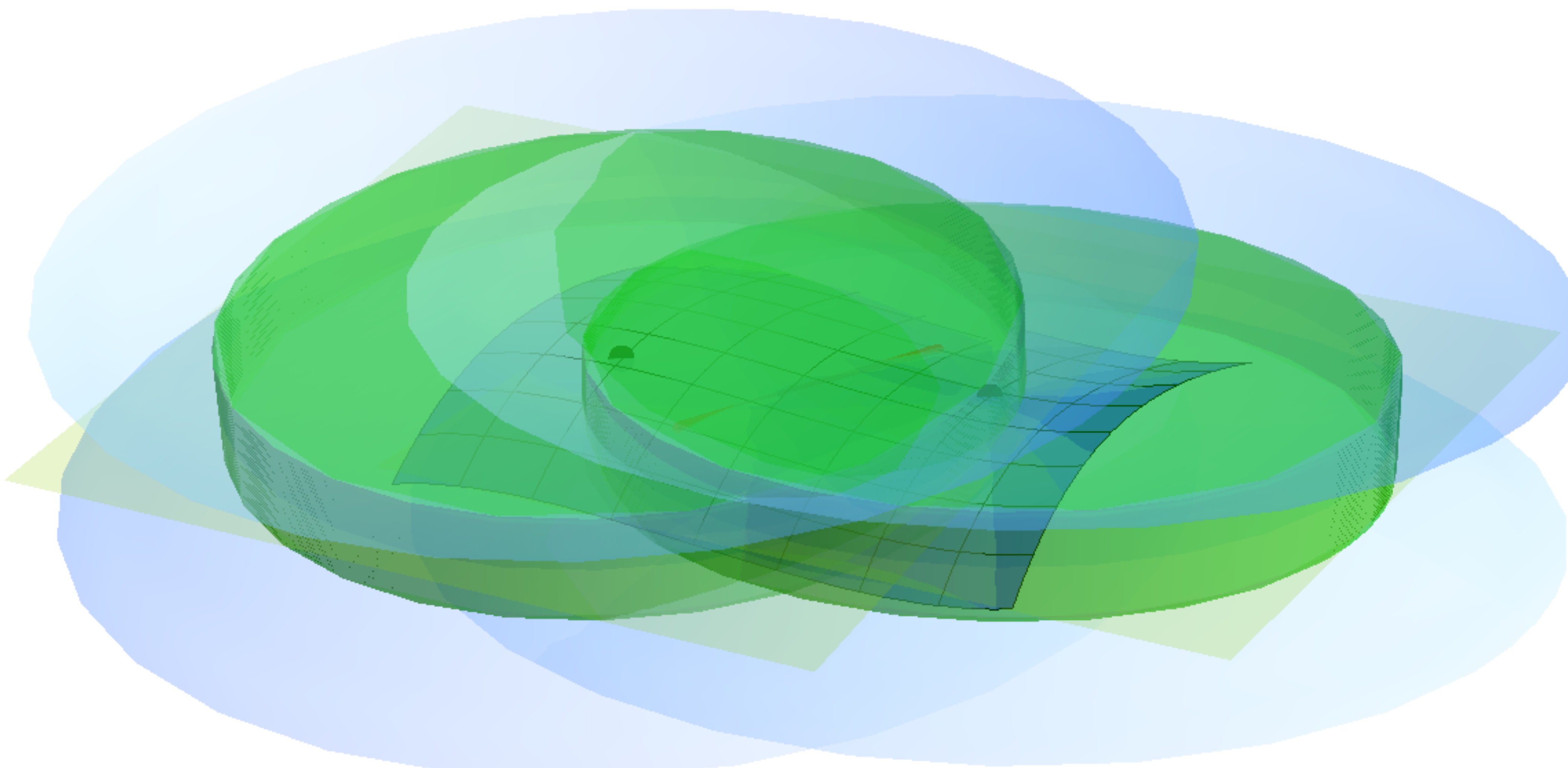


A pictorial proof



Thank you

Medial Axis and Reach

Medial axis of an **open set O:**
« set of points in O who have at least two closest points on the boundary of O »

Medial axis of a **closed set C**
= Medial axis of complement of C:
« set of points who have at least two closest points in C »

Reach of a **closed set C**
« infimum of distances between C and its medial axis»