Geometric methods for the conception of optical components in non-imaging optics

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Non-imaging optics

**Goal:** build optical components (mirrors or lenses) given

- a light *source* distribution
- a *prescribed* *target* distribution

Applications:

- Hydroponic agriculture
- Public lighting
- Car beams
- Caustic design
Non-imaging problems

- Source can be **collimated** or **punctual**
- Target can be at
  - infinity: directions in $\mathbb{S}^2 \rightarrow$ **Far-Field**
  - finite distance: points in $\mathbb{R}^3 \rightarrow$ **Near-Field**
Non-imaging problems

**Semi-discrete** setting:

- Source is a probability density $\rho$ on $X \subset \mathbb{R}^3$
- Target is a *discrete* measure $\sigma = \sum_{i=1}^{N} \sigma_i \delta_{y_i}$ supported on $Y \subset \mathbb{R}^3$
Outline

Part 1: General framework

▶ Setting 1: Collimated Source Mirror
▶ Setting 2: Collimated Source Lens
▶ Setting 3: Point Source Mirror
▶ Setting 4: Point Source Lens

Part 2: Damped Newton algorithm

▶ Description
▶ Convergence analysis
▶ Numerical results
General framework
The different settings
Setting 1: Collimated Source Mirror

**Setting:** \( X = \mathbb{R}^2 \times \{0\} \) and \( Y \subseteq S^2 \)

(Convex) parametrization of \( \mathcal{R} \):

\[
\mathcal{R}_\psi : x \in \mathbb{R}^2 \mapsto (x, \max_{1 \leq i \leq N} \langle x | p_i \rangle - \psi_i)
\]

where

- \( p_i \in \mathbb{R}^2 \) = slope of the plane that reflects the ray \((0,0,1)\) towards \( y_i \),
- \( \psi_i \in \mathbb{R} \) its elevation
Setting 1: Collimated Source Mirror

Visibility cell of \( y_i \): \( V_i(\psi) = \{ x \in \mathbb{R}^2 \times \{0\} | x \text{ reflected towards direction } y_i \} \)

We set \( G_i(\psi) = \int_{V_i(\psi)} \rho(x) dx \) and \( G(\psi) = (G_i(\psi))_{1 \leq i \leq N} \)

Collimated Source Mirror problem

Find \( \psi \in \mathbb{R}^N \) such that \( G(\psi) = \sigma \)
Setting 1: Collimated Source Mirror

Computation of $V_i(\psi)$

We recall $R_\psi(x) = (x, \max_i \langle x|p_i \rangle - \psi_i)$ and we have

$$V_i(\psi) = \{ x \in \mathbb{R}^2 | \forall j, -\langle x|p_i \rangle + \psi_i \leq -\langle x|p_j \rangle + \psi_j \}$$

$$= \{ x \in \mathbb{R}^2 | \forall j, c(x, y_i) + \psi_i \leq c(x, y_j) + \psi_j \}$$

$$=: \text{Lag}_i(\psi) \text{ for } c(x, y) = -\langle x|y \rangle$$

and

$$V_i(\psi) = (\mathbb{R}^2 \times \{0\}) \cap \text{Pow}_i(P)$$

where $p_i = \frac{p_{\mathbb{R}^2}(y_i-e_z)}{\langle y_i-e_z|e_z \rangle}$ and $\omega_i = 2\psi_i - \|p_i\|^2$
Setting 1: Collimated Source Mirror

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where $p_i = \frac{p_{R^2}(y_i - e_z)}{\langle y_i - e_z|e_z \rangle}$ and $\omega_i = 2\psi_i - \|p_i\|^2$

**Remark:** concave parametrization

$R_\psi(x) = (x, \min_i \langle x|p_i \rangle + \psi_i)$ and one can replace $p_i$ by its opposite in the previous expressions
Setting 2: Collimated Source Lens

**Setting:** \( X = \mathbb{R}^2 \times \{0\}, \ Y \subseteq \mathbb{S}_+^2 \) and \( \kappa = \text{ratio of the refractive indices} \)
Setting 2: Collimated Source Lens

**Setting:** \( X = \mathbb{R}^2 \times \{0\}, \ Y \subseteq \mathbb{S}^2_+ \) and \( \kappa = \text{ratio of the refractive indices} \)

**Convex parametrization**

\[
\mathcal{R}_\psi(x) = (x, \max_{1 \leq i \leq N} \langle x | p_i \rangle - \psi_i)
\]

and

\[
V_i(\psi) = \{ x \in \mathbb{R}^2 \times \{0\} \mid \text{refracted towards } y_i \} =: \text{Lag}_i(\psi) \text{ for } c(x, y_i) = -\langle x | p_i \rangle
\]

\[
V_i(\psi) = (\mathbb{R}^2 \times \{0\}) \cap \text{Pow}_i(P)
\]

where \( p_i = -\frac{p_{\mathbb{R}^2}(y_i - \kappa e_z)}{\langle y_i - \kappa e_z | e_z \rangle} \) and \( \omega_i = 2\psi_i - \|p_i\|^2 \)
Setting 2: Collimated Source Lens

Setting: $X = \mathbb{R}^2 \times \{0\}$, $Y \subseteq S^2_+$ and $\kappa =$ ratio of the refractive indices

Convex parametrization

$$\mathcal{R}_\psi(x) = (x, \max_{1 \leq i \leq N} \langle x | p_i \rangle - \psi_i)$$

and

$$V_i(\psi) = \{ x \in \mathbb{R}^2 \times \{0\} | \times \text{refracted towards } y_i \}$$

$$= : \text{Lag}_i(\psi) \text{ for } c(x, y_i) = -\langle x | p_i \rangle$$

$$V_i(\psi) = (\mathbb{R}^2 \times \{0\}) \cap \text{Pow}_i(P)$$

where $p_i = -p_{\mathbb{R}^2} (y_i - \kappa e_z) \over \langle y_i - \kappa e_z | e_z \rangle$ and $\omega_i = 2\psi_i - \| p_i \|^2$

Remark: concave parametrization

$$\mathcal{R}_{\psi}(x) = (x, \min_i \langle x | p_i \rangle + \psi_i)$$
Setting 3: Point Source Mirror

**Setting:** $X = \mathbb{S}^2$ and $Y \subseteq \mathbb{S}^2$
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Parametrization: intersection of confocal paraboloids $\implies$ convex

$$
\mathcal{R}_\psi : x \in \mathbb{S}^2 \mapsto \max_{1 \leq i \leq N} \frac{\psi_i}{1 - \langle x | y_i \rangle}
$$

where $\psi_i$ is the focal distance of the $i$-th paraboloid, and we have

$$
V_i(\psi) = \mathbb{S}^2 \cap \text{Pow}_i(P)
$$
**Setting 3: Point Source Mirror**

**Setting:** \( X = \mathbb{S}^2 \) and \( Y \subseteq \mathbb{S}^2 \)

**Parametrization:** intersection of confocal paraboloids \( \implies \) convex

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\mathcal{R}_\psi : x \in \mathbb{S}^2 \mapsto \max_{1 \leq i \leq N} \frac{\psi_i}{1 - \langle x | y_i \rangle}
\]

where \( \psi_i \) is the focal distance of the \( i \)-th paraboloid, and we have

\[
V_i(\psi) = \mathbb{S}^2 \cap \text{Pow}_i(P)
\]

**Remark:** union of confocal paraboloids
Setting 4: Point Source Lens

**Setting:** \( X = S^2, Y \subseteq S^2_+ \) and \( \kappa < 1 \)

**Parametrization:** intersection of confocal ellipsoids \( \Rightarrow \) convex

\[
\mathcal{R}_\psi : x \in S^2 \mapsto \max_{1 \leq i \leq N} \frac{\psi_i}{1 - \kappa \langle x | y_i \rangle}
\]

where \( \psi_i \) is one of the focal distances of the \( i \)-th ellipsoid, and

\[
V_i(\psi) = S^2 \cap \text{Pow}_i(P)
\]

**Remark:** union of confocal ellipsoids
Common structure

Semi discrete Monge Ampère equation

Find $\psi \in \mathbb{R}^N$ such that $G(\psi) = \sigma$
Optimal transport as Concave Maximization

**Common structure**

**Semi discrete Monge Ampère equation**

| Find $\psi \in \mathbb{R}^N$ such that $G(\psi) = \sigma$ |

**Efficient evaluation of $G$**

- Visibility cells: $V_i(\psi) = X \cap \text{Pow}_i(P)$
- Collimated: $X = \mathbb{R}^2 \times \{0\}$
- Punctual: $X = S^2$
Common structure

Semi discrete Monge Ampère equation

Find $\psi \in \mathbb{R}^N$ such that $G(\psi) = \sigma$

Efficient evaluation of $G$

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- Collimated: $X = \mathbb{R}^2 \times \{0\}$
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Damped Newton algorithm: convergence results known for

- Quadratic cost in the plane [Mirebeau, 2015]
- Many cost functions (MTW) [Kitagawa et al., 2016]
Common structure

Semi discrete Monge Ampère equation

Find $\psi \in \mathbb{R}^N$ such that $G(\psi) = \sigma$

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Damped Newton algorithm: convergence results known for

- Quadratic cost in the plane [Mirebeau, 2015]
- Many cost functions (MTW) [Kitagawa et al., 2016]

$\implies$ We will use the same algorithm
Generic algorithm
Overview

**Algorithm**: Mirror / lens construction

**Input**
- A light source intensity function $\rho_{in}$
- A target light intensity function $\sigma_{in}$
- A tolerance $\eta > 0$

**Output**
- A triangulation $\mathcal{R}_T$ of a mirror or lens $\mathcal{R}$
Overview

**Algorithm:** Mirror / lens construction

**Input**
- A light source intensity function $\rho_{in}$
- A target light intensity function $\sigma_{in}$
- A tolerance $\eta > 0$

**Output**
- A triangulation $\mathcal{R}_T$ of a mirror or lens $\mathcal{R}$

**Step 1** Initialization

\[
T, \rho \leftarrow \text{DISCRETIZATION\_SOURCE}(\rho_{in}) \\
Y, \sigma \leftarrow \text{DISCRETIZATION\_TARGET}(\sigma_{in}) \\
\psi^0 \leftarrow \text{INITIAL\_WEIGHTS}(Y)
\]
Overview

**Algorithm:** Mirror / lens construction

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- A light source intensity function $\rho_{in}$
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**Step 1** Initialization

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$\psi^0 \leftarrow \text{INITIAL\_WEIGHTS}(Y)$

**Step 2** Solve $G(\psi) = \sigma$

$\psi \leftarrow \text{DAMPED\_NEWTON}(T, \rho, Y, \sigma, \psi^0, \eta)$
Overview

Algorithm: Mirror / lens construction

Input
A light source intensity function $\rho_{in}$
A target light intensity function $\sigma_{in}$
A tolerance $\eta > 0$

Output
A triangulation $\mathcal{R}_T$ of a mirror or lens $\mathcal{R}$

Step 1 Initialization

$T, \rho \leftarrow \text{DISCRETIZATION\_SOURCE}(\rho_{in})$
$Y, \sigma \leftarrow \text{DISCRETIZATION\_TARGET}(\sigma_{in})$
$\psi^0 \leftarrow \text{INITIAL\_WEIGHTS}(Y)$

Step 2 Solve $G(\psi) = \sigma$

$\psi \leftarrow \text{DAMPED\_NEWTON}(T, \rho, Y, \sigma, \psi^0, \eta)$

Step 3 Construct a triangulation $\mathcal{R}_T$ of $\mathcal{R}$

$\mathcal{R}_T \leftarrow \text{SURFACE\_CONSTRUCTION}(\psi, \mathcal{R}_\psi)$
Step 1: Initialization

Discretization:

- $\rho_{in}$ is discretized by a piecewise affine function $\rho$ on a triangulation $T$
- $\sigma_{in}$ is discretized by a discrete measure $\sigma$ on a 3D point cloud $Y$
Step 1: Initialization

Discretization:
- $\rho_{in}$ is discretized by a piecewise affine function $\rho$ on a triangulation $T$
- $\sigma_{in}$ is discretized by a discrete measure $\sigma$ on a 3D point cloud $Y$

Initial weights: we must ensure that at each step $\forall i, G_i(\psi) > 0$

\[
\frac{\partial G_i}{\partial \psi_j}(\psi) \propto \int_{V_{i,j}(\psi)} \rho(x) dx
\]
Step 1: Initialization

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Initial weights: we must ensure that at each step $\forall i, G_i(\psi) > 0$

$$\frac{\partial G_i}{\partial \psi_j} (\psi) \propto \int_{V_{i,j}(\psi)} \rho(x) dx$$

$$\implies \frac{\partial G_i}{\partial \psi_j} \text{ is not continuous!}$$
Step 1: Initialization

Discretization:

- $\rho_{in}$ is discretized by a piecewise affine function $\rho$ on a triangulation $T$
- $\sigma_{in}$ is discretized by a discrete measure $\sigma$ on a 3D point cloud $Y$

Initial weights: we must ensure that at each step $\forall i, G_i(\psi) > 0$

At the beginning:

- simple settings $\implies$ easy choices can be made
- more complex configurations like pillows $\implies$ other strategies
Step 2: Damped Newton algorithm

**Algorithm:** Mirror / lens construction

**Input**
- A function $\rho : T \to \mathbb{R}^+$
- A discrete measure $\sigma = (\sigma_i)_{1 \leq i \leq N}$ supported on $Y$
- An initial vector of weights $\psi^0$
- A tolerance $\eta > 0$

**Output**
- A vector $\psi \in \mathbb{R}^N$

**Step 1** Transformation into an optimal transport problem

- If $X = \mathbb{R}^2 \times \{0\}$, then $\tilde{\psi}^0 = \psi^0$ and $\tilde{G}_i = G_i$
- If $X = \mathbb{S}^2$, then $\tilde{\psi}^0 = \ln(\psi^0)$ and $\tilde{G}_i = G_i \circ \exp$
Step 2: Damped Newton algorithm

Algorithm: Mirror / lens construction

Step 2 Solve $\tilde{G}(\tilde{\psi}) = \sigma$

Initialization $\epsilon_0 = \min[\min_i G_i(\psi^0), \min_i \sigma_i]$

$k = 0$

While $\|\tilde{G}(\tilde{\psi}^k) - \sigma\|_\infty > \eta$

- Compute $d_k = -D\tilde{G}(\tilde{\psi}^k) + (\tilde{G}(\tilde{\psi}^k) - \sigma)$

- Determine the minimum $\ell \in \mathbb{N}$ such that $\tilde{\psi}^{k,\ell} := \tilde{\psi}^k + 2^{-\ell}d_k$ satisfies:

\[
\begin{cases}
\min_i \tilde{G}_i(\tilde{\psi}^{k,\ell}) \geq \epsilon_0 \\
\|\tilde{G}(\tilde{\psi}^{k,\ell}) - \sigma\|_\infty \leq (1 - 2^{-(\ell+1)})\|\tilde{G}(\tilde{\psi}^k) - \sigma\|_\infty
\end{cases}
\]

- Set $\tilde{\psi}^{k+1} = \tilde{\psi}^k + 2^{-\ell}d_k$ and $k \leftarrow k + 1$

Return $\psi := \tilde{\psi}^k$ if $X = \mathbb{R}^2 \times \{0\}$ or $\psi := \exp(\tilde{\psi}^k)$ if $X = \mathbb{S}^2$
Step 2: Damped Newton algorithm

**Computation of** $DG$: automatic differentiation $\implies$ genericity
Step 2: Damped Newton algorithm

Computation of $DG$: automatic differentiation $\implies$ genericity

**Theorem ([Mérigot, M., Thibert, 2017])**

Assume $\rho$ is a regular simplicial measure and that the points $p_1, \ldots, p_N$ are in generic position. Then:

- $\tilde{G}$ is of class $C^1$ on $\mathbb{R}^N$,
- $\tilde{G}$ is strictly monotone in the sense

$$\forall \psi \in K^+, \forall v \in \{cst\}^\perp \setminus \{0\}, \langle D\tilde{G}(\psi)v|v\rangle < 0$$

$\implies$ the proposed damped Newton algorithm converges in a finite number of steps and

$$\|\tilde{G}(\psi^{k+1}) - \sigma\|_\infty \leq \left(1 - \frac{\tau^*}{2}\right) \|\tilde{G}(\psi^k) - \sigma\|_\infty$$

where $\tau^* \in ]0, 1]$ depends on $\rho, \sigma$ and $\epsilon_0$. 

Step 2: Damped Newton algorithm

**Computation of** $DG$: automatic differentiation $\implies$ genericity

**Convergence** known for:

- Collimated settings: mirror and lens (convex and concave)
- Punctual settings: mirror (intersection), lens (union)
Step 3: Surface construction

$\mathcal{R}_T$ is the *lifted* triangulation *dual* to the *Visibility* diagram

**Collimated Source Mirror** (convex)
Numerical results
General framework
General framework
Collimated source

Target / Mean curvature / Forward simulation

Convex Collimated Source Mirror
problem with a uniform light source

Concave Collimated Source Lens
problem with a uniform light source
Punctual source

Target / Mean curvature / Forward simulation

Concave Point Source Mirror
problem with a uniform light source

Point Source Lens problem with a uniform light source
Non uniform source

Source

$\mathcal{R}_T$

$\mathcal{R}_T$ for a uniform source
Pillows

**Goal:** decompose the optical component into several smaller *pillows*
Goal: decompose the optical component into several smaller *pillows*

Problem: support of the pillow is *small* \(\Rightarrow\) choice of the initial weights

Solution: interpolation between two source densities
Pillows

**Interpolation**: pillow = left part of the plane

$$V_i(\psi) \cap P = \emptyset$$
Pillows

Interpolation: pillow = left part of the plane
Pillows

**Interpolation:** pillow = left part of the plane
Pillows

Example: lens made of 9 pillows, without and with an obstacle
Pillows

Without an obstacle
Pillows

With an obstacle
Demo
Conclusion & Perspectives

We saw

- a general framework to solve 8 different optical component design problems,
- and a generic and efficient algorithm able to solve them.

**Code**: OT between a density supported on a triangulated surface and a discrete measure on a 3D point cloud for the quadratic cost
Conclusion & Perspectives

We saw

▶ a *general framework* to solve 8 different optical component design problems,
▶ and a *generic* and *efficient* algorithm able to solve them.

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**Perspectives:**

▶ Near-Field: optimal transport replaced by a *prescribed Jacobian*
▶ extended sources
▶ initialization strategies
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Thank you for your attention.
Conclusion & Perspectives

We saw

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Perspectives:

- Near-Field: optimal transport replaced by a *prescribed Jacobian*
- extended sources
- initialization strategies

Thank you for your attention.
Pillows (bis)

**Image alignment**: we can not use the Far-Field assumption

we use an iterative method to simulate a *Near-Field* target