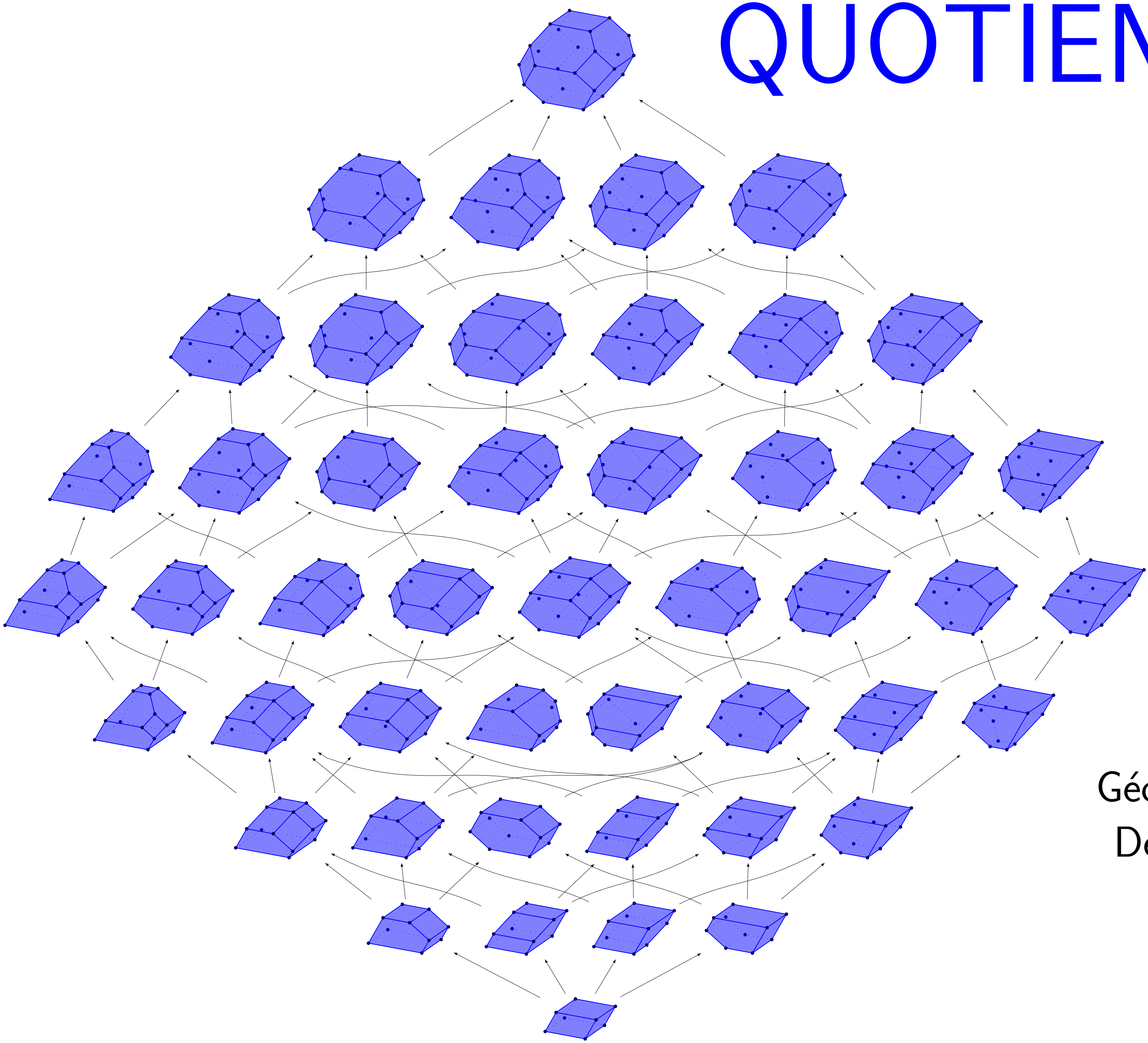

QUOTIENTOPES

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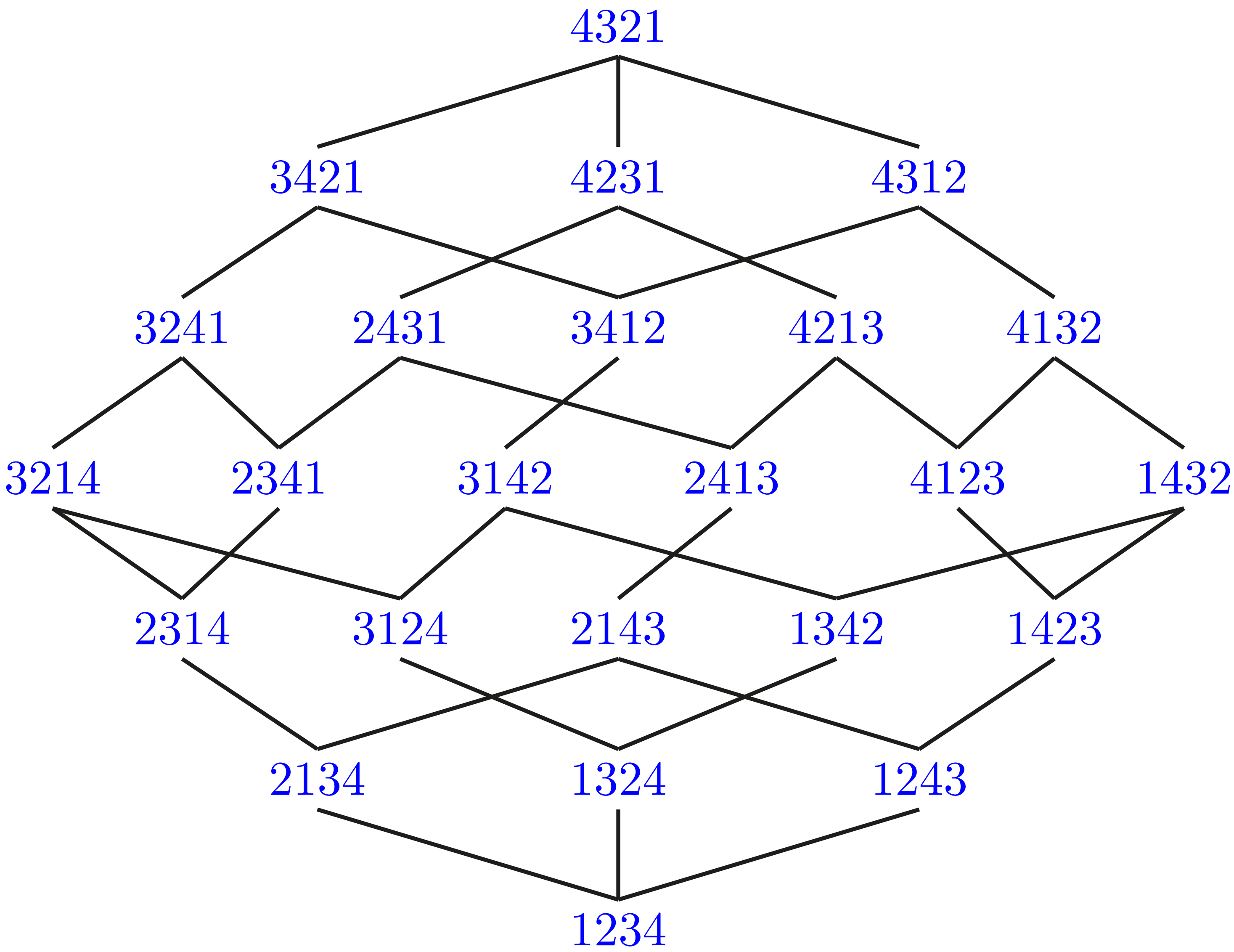
Journées de
Géométrie Algorithmique
December 11–15, 2017

WEAK ORDER & PERMUTAHEDRON

WEAK ORDER

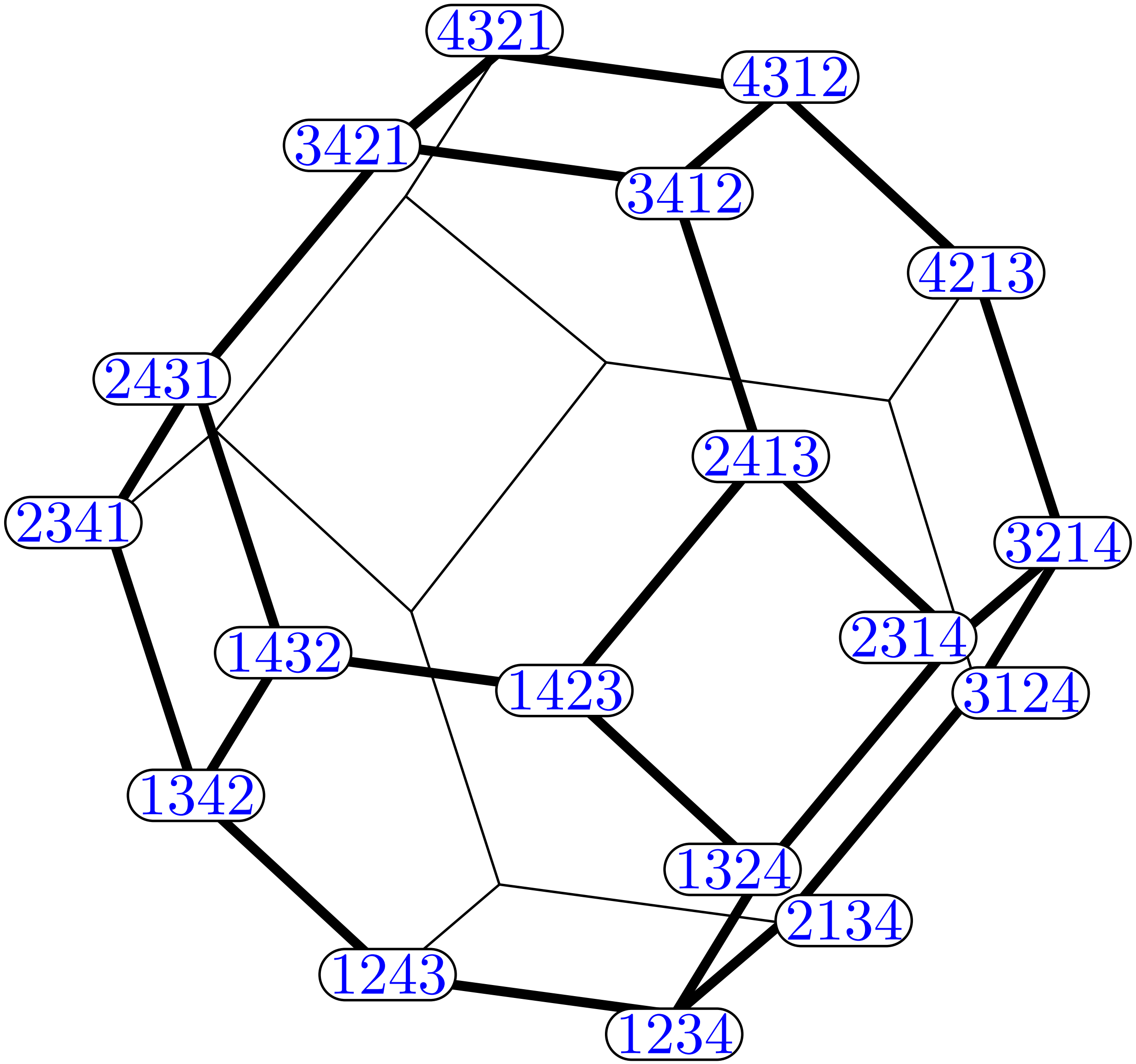
inversions of $\sigma \in \mathfrak{S}_n$ = pair (σ_i, σ_j) such that $i < j$ and $\sigma_i > \sigma_j$

weak order = permutations of $\mathfrak{S}_n$ ordered by inclusion of inversion sets



PERMUTAHEDRON

Permutohedron $\text{Perm}(n) = \text{conv}\{(\sigma(1), \ldots, \sigma(n)) \in \mathbb{R}^n \mid \sigma \in \mathfrak{S}_n\}$



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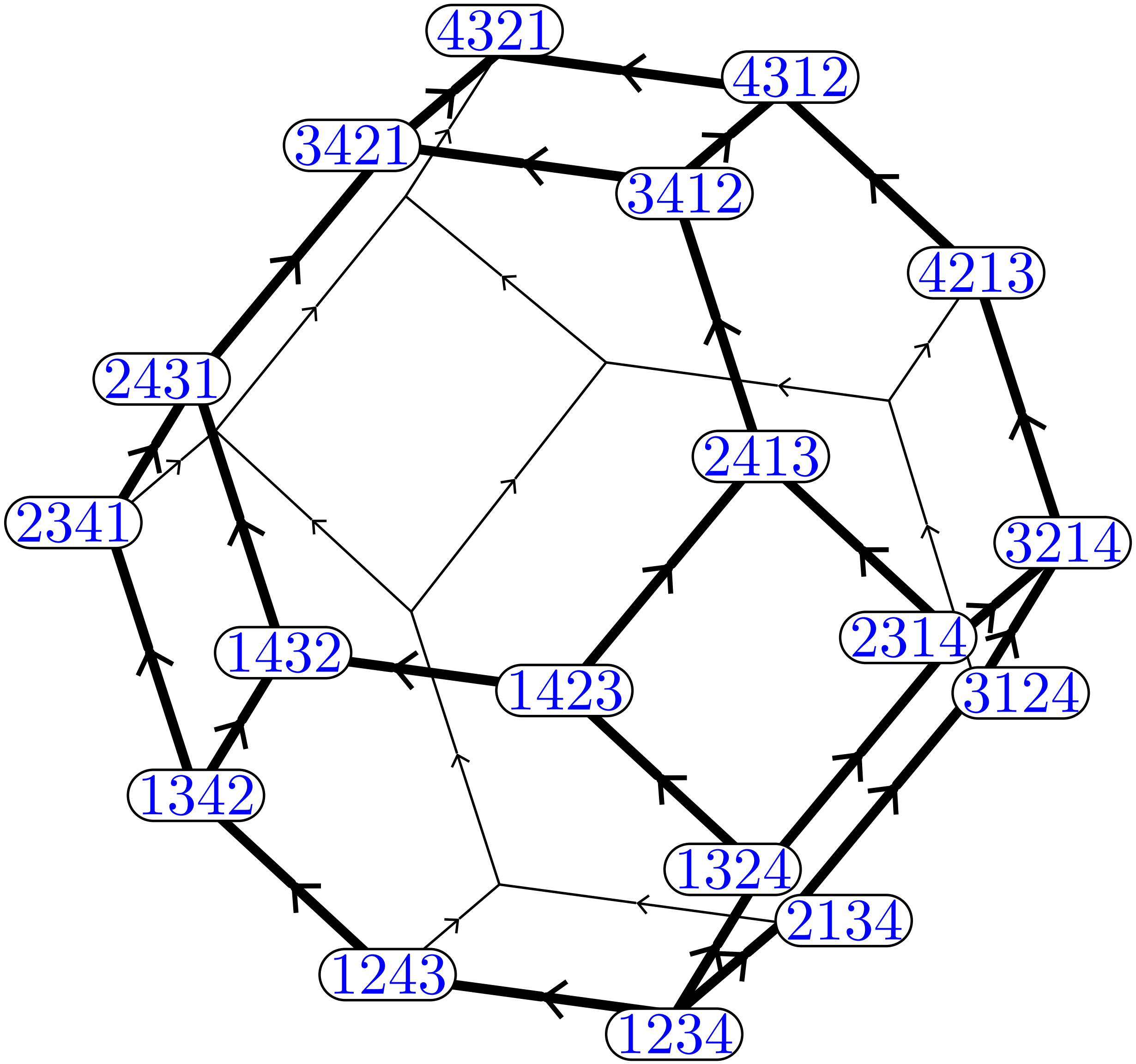
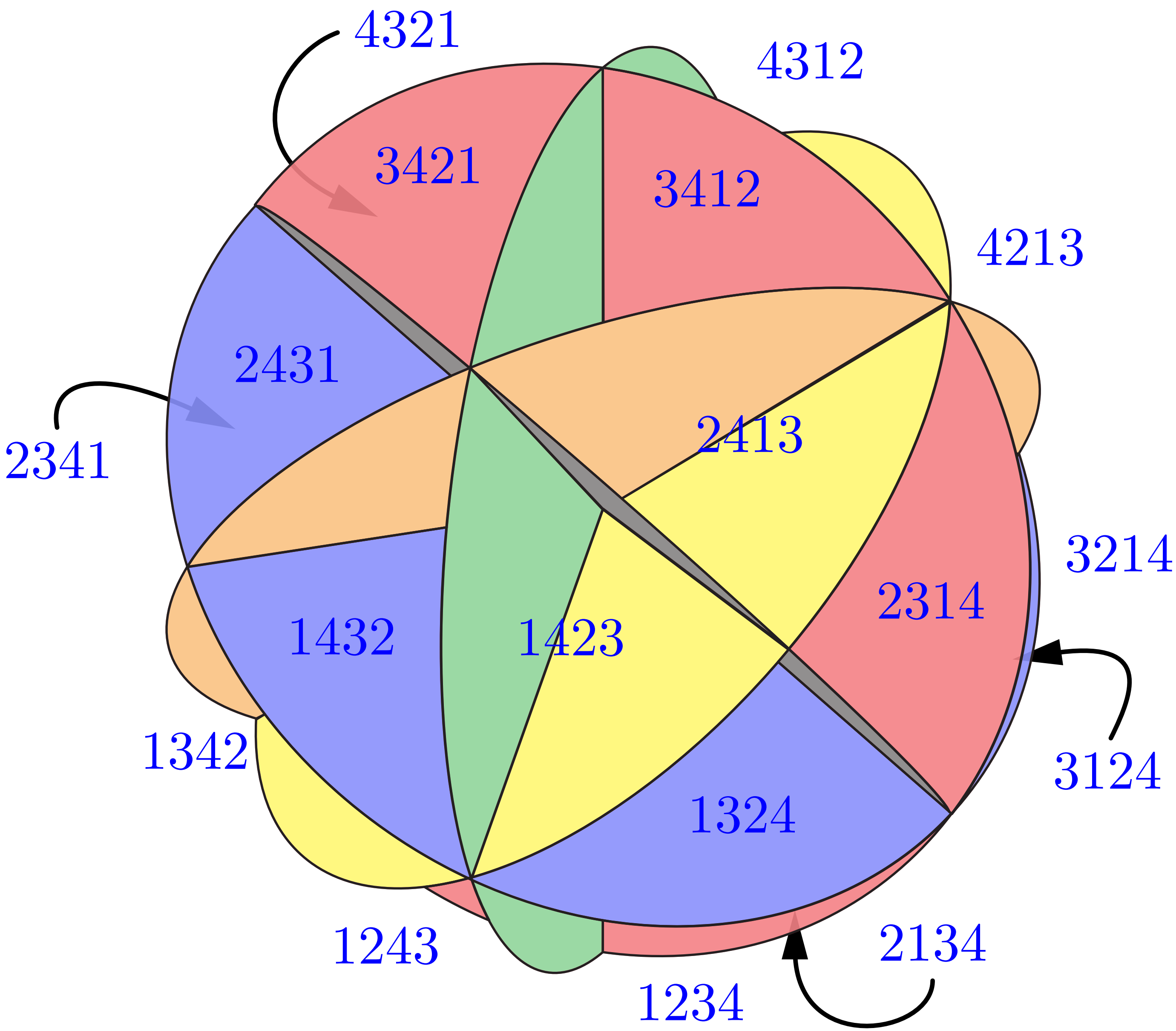


weak order = orientation of the graph of $\mathsf{Perm}(n)$

COXETER ARRANGEMENT

Coxeter fan = fan defined by the hyperplane arrangement $\{\mathbf{x} \in \mathbb{R}^n \mid x_i = x_j\}_{1 \le i < j \le n}$



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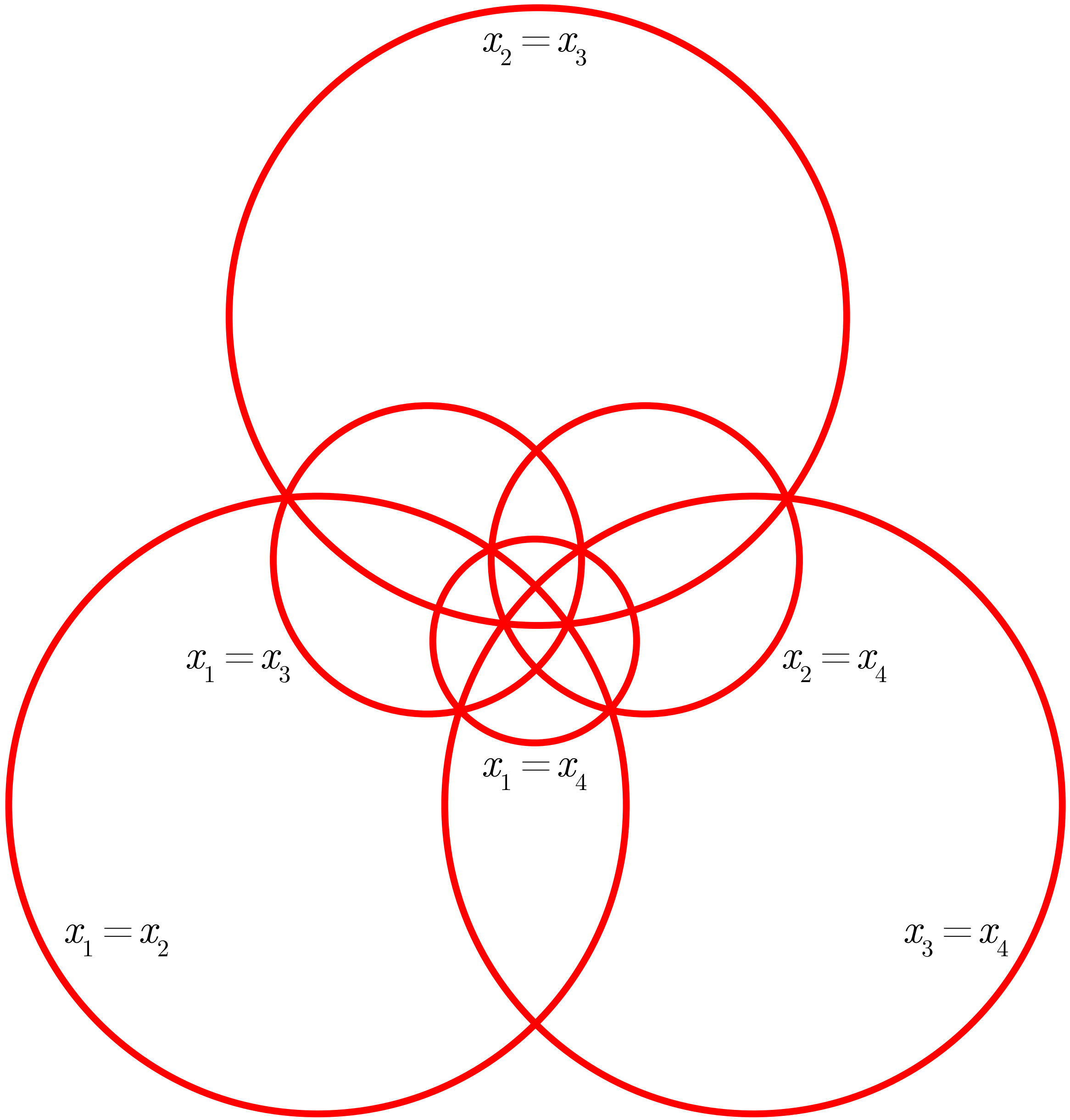


LATTICE QUOTIENTS

Reading, *Lattice congruences, fans and Hopf algebras* ('05)
Reading, *Finite Coxeter groups and the weak order* ('16)
Reading, *Lattice theory of the poset of regions* ('16)

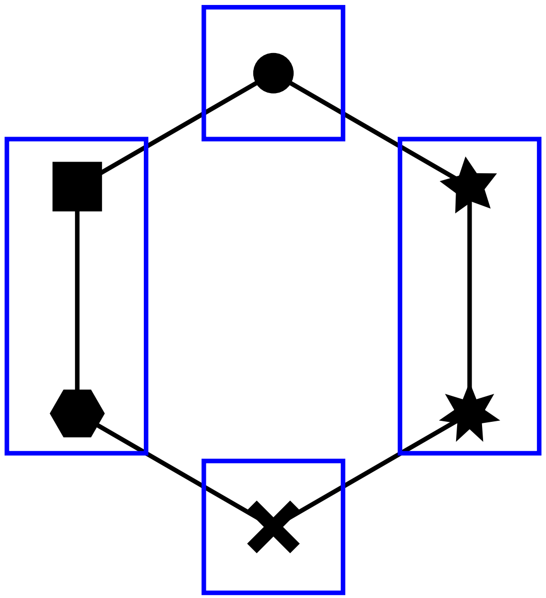
LATTICE CONGRUENCES

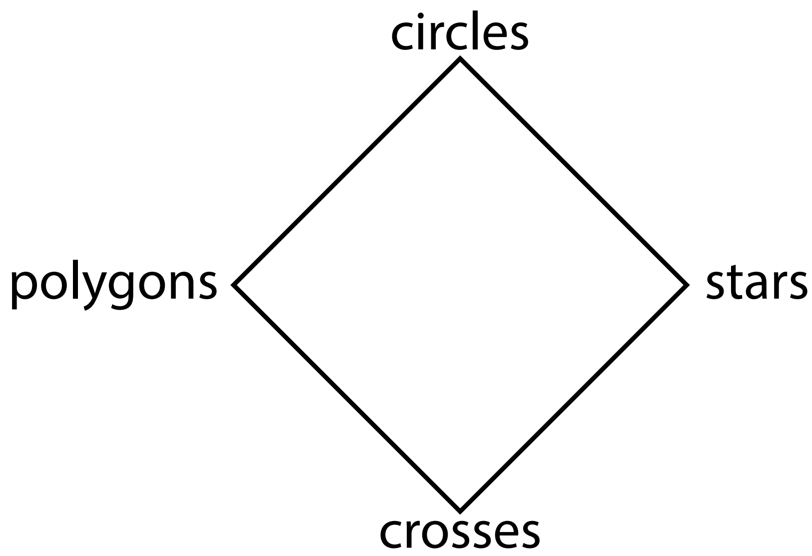
lattice congruence = equiv. rel. $\equiv$ on L which respects meets and joins

$$x \equiv x' \quad \text{and} \quad y \equiv y' \qquad \Longrightarrow \qquad x \wedge y \equiv x' \wedge y' \quad \text{and} \quad x \vee y \equiv x' \vee y'$$

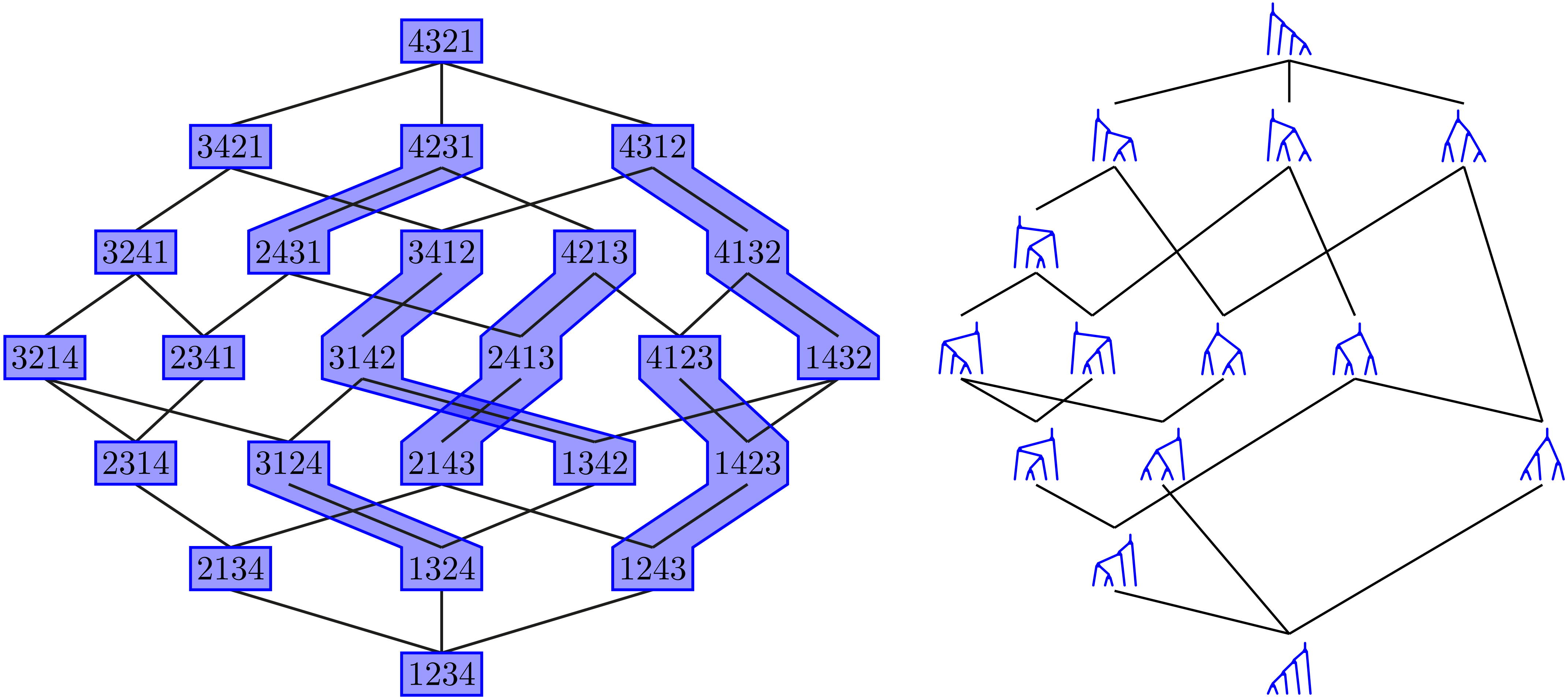
lattice quotient of $L/\equiv$ = lattice on equiv. classes of L under $\equiv$ where

- $X \le Y \iff \exists x \in X,\ y \in Y,\quad x \le y$
- $X \wedge Y$ = equiv. class of $x \wedge y$ for any $x \in X$ and $y \in Y$
- $X \vee Y$ = equiv. class of $x \vee y$ for any $x \in X$ and $y \in Y$





EXM: TAMARI LATTICE

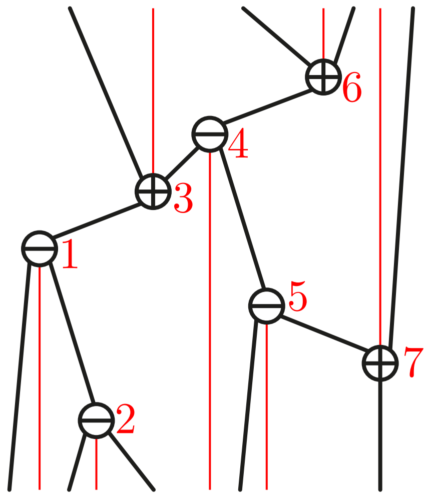


Tamari lattice = lattice quotient of the weak order by the relation "same binary tree"

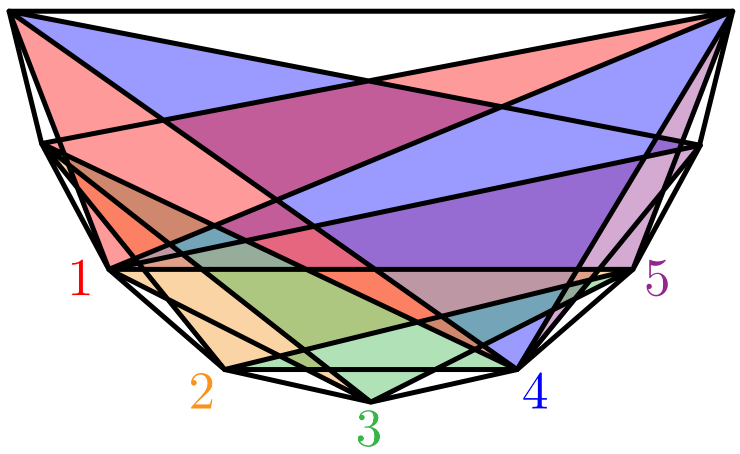
Catalan combinatorics — Associahedron — Non-crossing partitions — ...

RELEVANT LATTICE QUOTIENTS OF THE WEAK ORDER

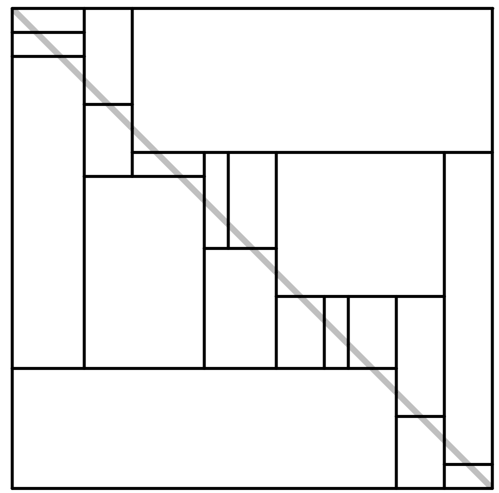
Cambrian trees



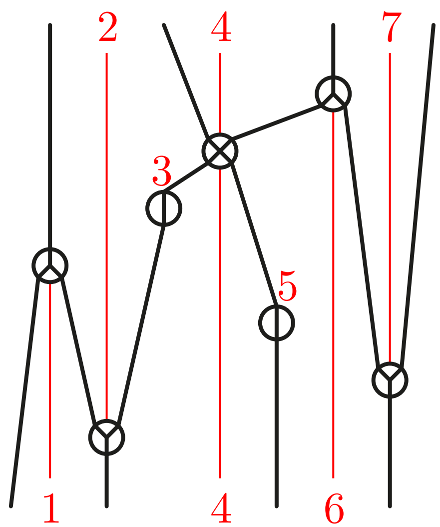
Acyclic k-triangulations



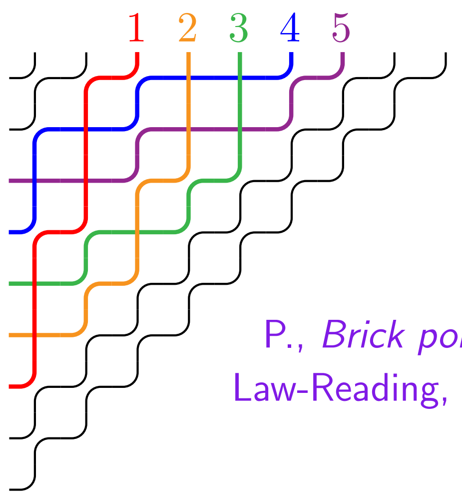
diagonal rectangulations



Permutrees



Pipe dreams



Reading, *Cambrian lattices* ('06)

Chatel-P., *Cambrian algebra* ('17)

P., *Brick polytopes, lattice quotients, and Hopf algebras* ('15$^+$)

Law-Reading, *The Hopf algebra of diagonal rectangulations* ('12)

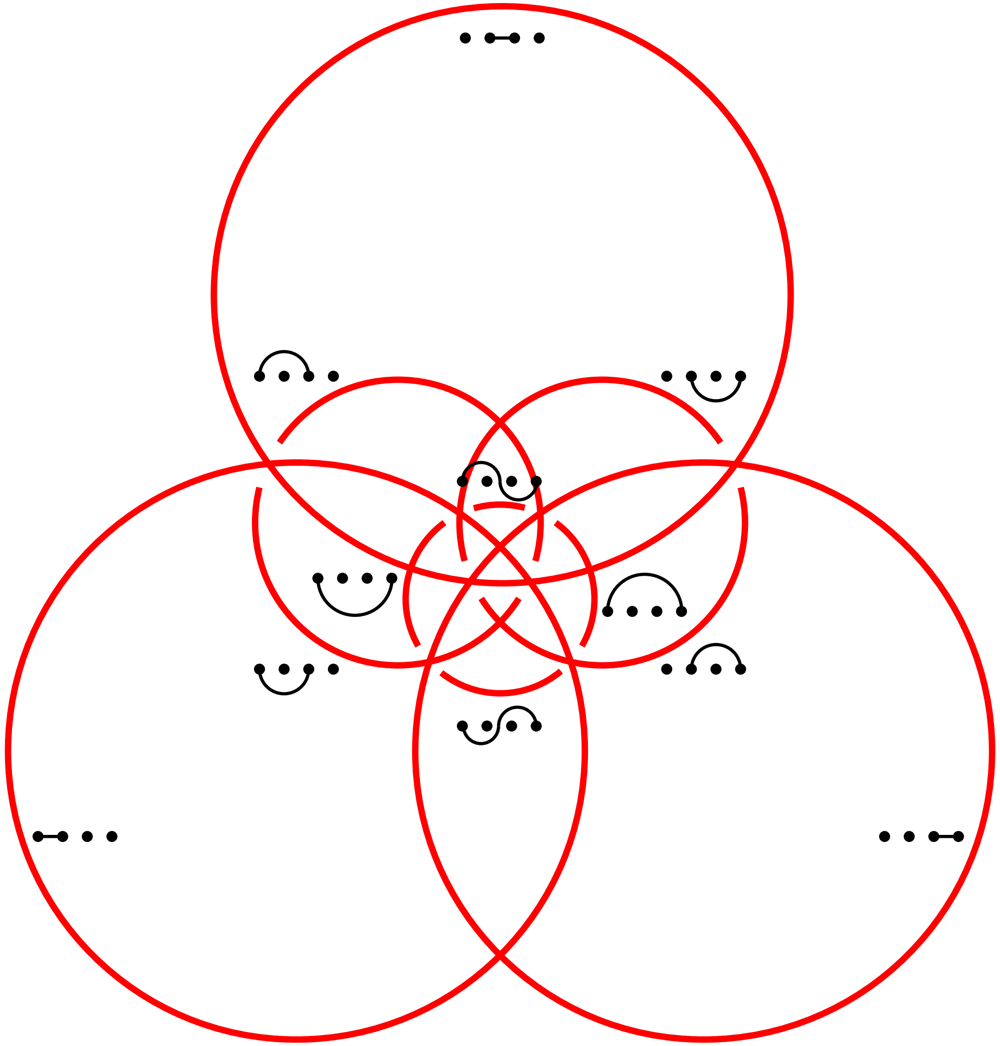
P.-Pons, *Permutrees* ('17)

QUOTIENT FAN

Reading, *Lattice congruences, fans and Hopf algebras* ('05)
Reading, *Finite Coxeter groups and the weak order* ('16)
Reading, *Lattice theory of the poset of regions* ('16)

SHARDS

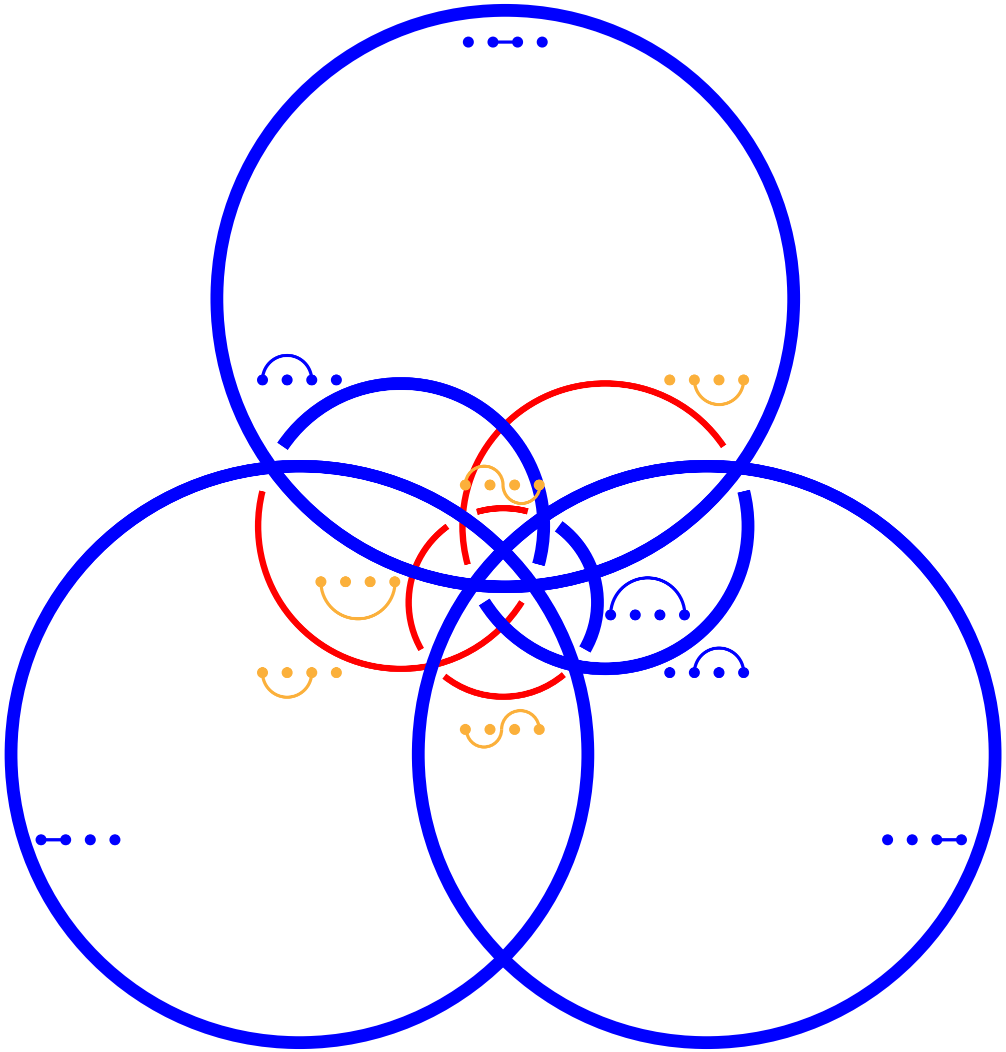
<u>shard</u> $\Sigma(i,j,n,S) := \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; x_i = x_j \text{ and } \left[\begin{array}{l} x_i \le x_k \text{ for all } k \in S \text{ while} \\ x_i \ge x_k \text{ for all } k \in]i,j[\smallsetminus S \end{array} \right. \right\}$



THM. The shards $\Sigma(i,j,n,S)$ for all subsets $S \subseteq]i,j[$ decompose the hyperplane $x_i = x_j$ into 2^{j-i-1} pieces.

SHARDS AND QUOTIENT FAN

shard $\Sigma(i,j,n,S) := \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; x_i = x_j \text{ and } \begin{bmatrix} x_i \le x_k \text{ for all } k \in S \text{ while} \\ x_i \ge x_k \text{ for all } k \in \,]i,j[\, \smallsetminus S \end{bmatrix} \right\}$



THM. For a lattice congruence $\equiv$ on $\mathfrak{S}_n$, the cones obtained by glueing the Coxeter regions of the permutations in the same congruence class of $\equiv$ form a fan $\mathcal{F}_\equiv$ of $\mathbb{R}^n$ whose dual graph realizes the lattice quotient $\mathfrak{S}_n/\equiv$.

Reading, *Lattice congruences, fans and Hopf algebras* ('05)

THM. Each lattice congruence $\equiv$ on $\mathfrak{S}_n$ corresponds to a set of shards $\mathbf{\Sigma}_\equiv$ such that the cones of $\mathcal{F}_\equiv$ are the connected components of the complement of the union of the shards in $\mathbf{\Sigma}_\equiv$.

Reading, *Lattice congruences, fans and Hopf algebras* ('05)

SHARD IDEALS

SHARD IDEALS

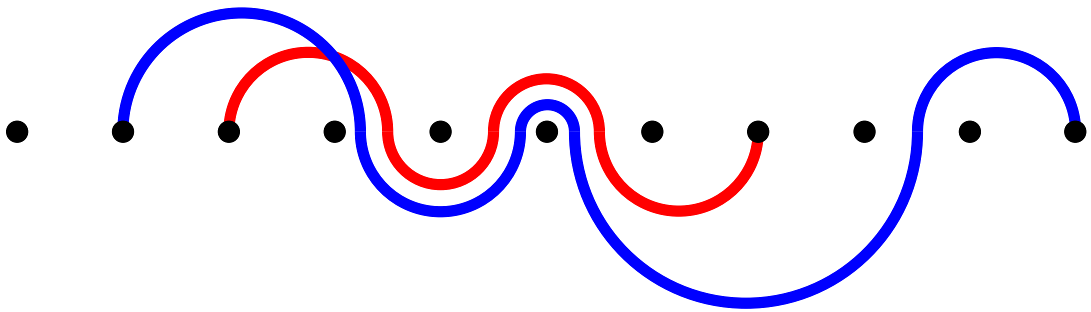
THM. Each lattice congruence $\equiv$ on $\mathfrak{S}_n$ corresponds to a set of shards $\mathbf{\Sigma}_{\equiv}$ such that the cones of $\mathcal{F}_{\equiv}$ are the connected components of the complement of the union of the shards in $\mathbf{\Sigma}_{\equiv}$.

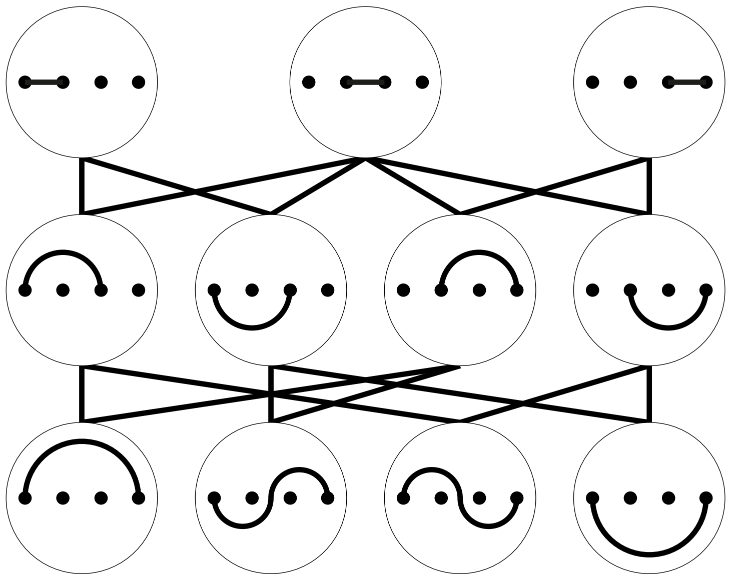
Reading, *Lattice congruences, fans and Hopf algebras* ('05)

THM. The following are equivalent for a set of shards $\mathbf{\Sigma}$:

- there exists a lattice congruence $\equiv$ on $\mathfrak{S}_n$ with $\mathbf{\Sigma} = \mathbf{\Sigma}_{\equiv}$,
- $\mathbf{\Sigma}$ is an upper ideal for the order $\Sigma(a, d, n, S) \prec \Sigma(b, c, n, T) \iff a \le b < c \le d$ and $T = S \cap \,]b, c[$.

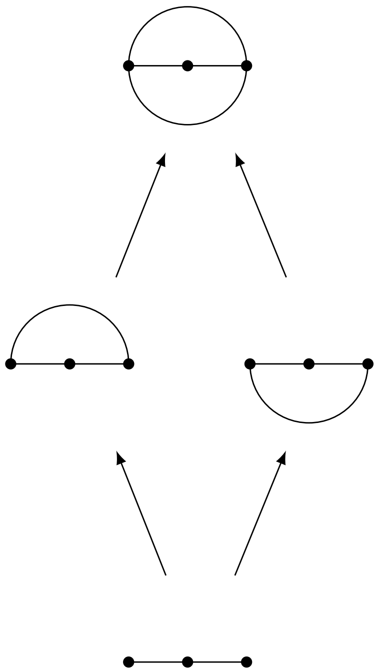
Reading, *Noncrossing arc diagrams and can. join representations* ('15)

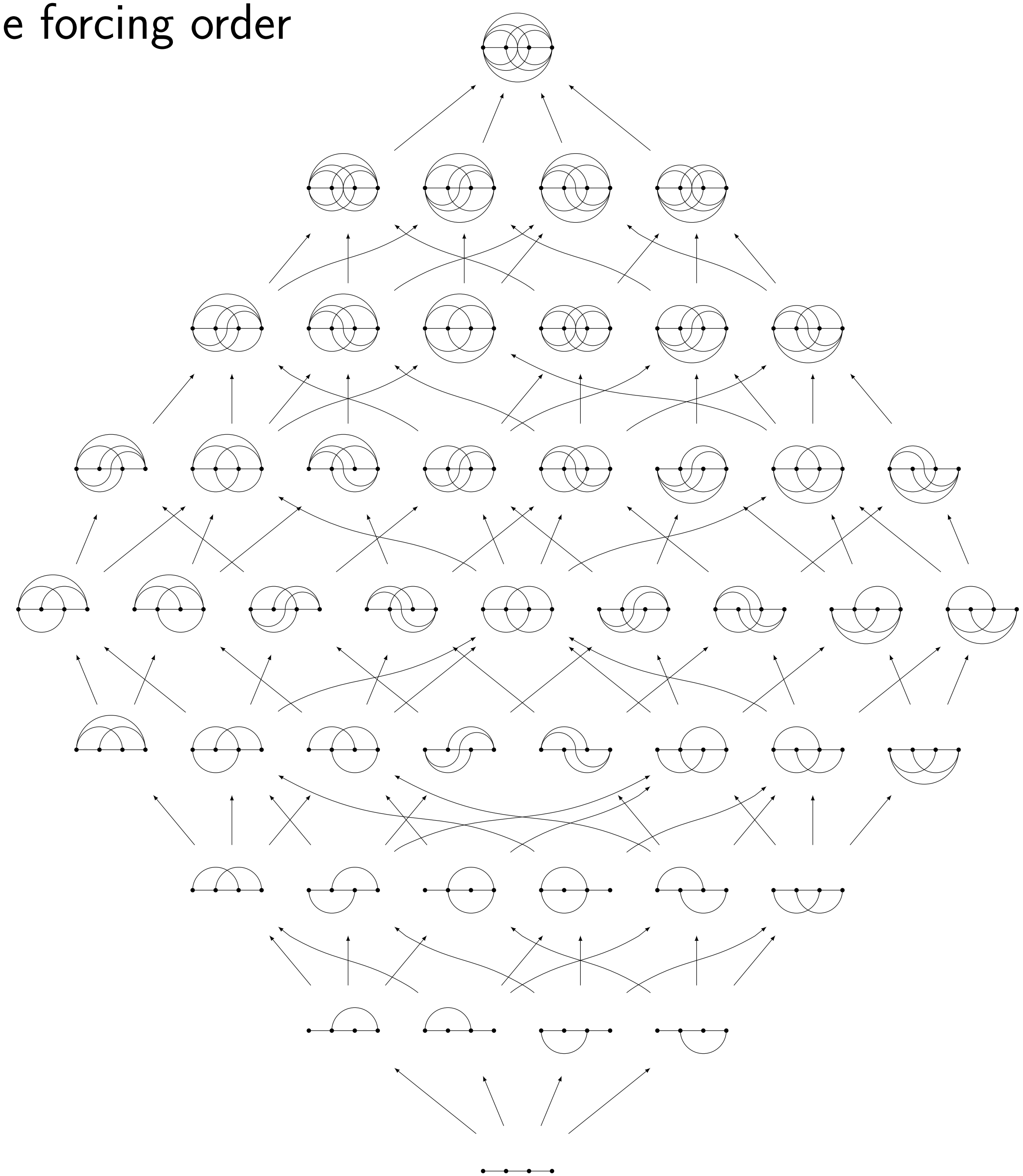




SHARD IDEALS

shard ideal = upper ideal for the forcing order





QUOTIENTOPES

QUOTIENTOPE

fix a forcing dominant function $f : \sigma \to \mathbb{R}_{>\mathbf{0}}$ ie. st. $f(\Sigma) > \sum_{\Sigma' \succ \Sigma} f(\Sigma')$ for any shard Σ.

for a shard $\Sigma = (i, j, n, S)$ and a subset $\varnothing \neq R \subsetneq [n]$ define the contribution

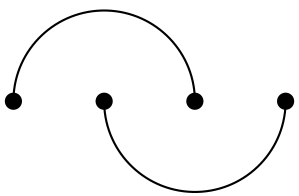
$$\gamma(\Sigma, R) := \begin{cases} 1 & \text{if } |R \cap \{i,j\}| = 1 \text{ and } S = R \cap \,]i,j[, \\ 0 & \text{otherwise} \end{cases}$$

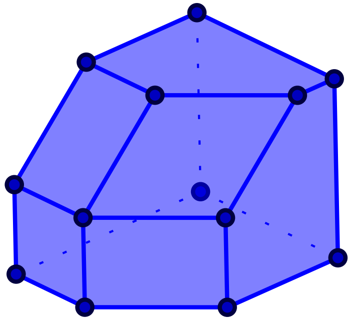
define height function h for $\varnothing \neq R \subsetneq [n]$ by $h^f_\equiv(R) := \sum_{\Sigma \in \mathbf{\Sigma}_\equiv} f(\Sigma)\, \gamma(\Sigma, R)$.

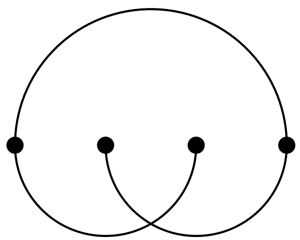
THM. For a lattice congruence $\equiv$ on $\mathfrak{S}_n$ and a forcing dominant function $f : \mathbf{\Sigma} \to \mathbb{R}_{>\mathbf{0}}$, the quotient fan $\mathcal{F}_\equiv$ is the normal fan of the polytope

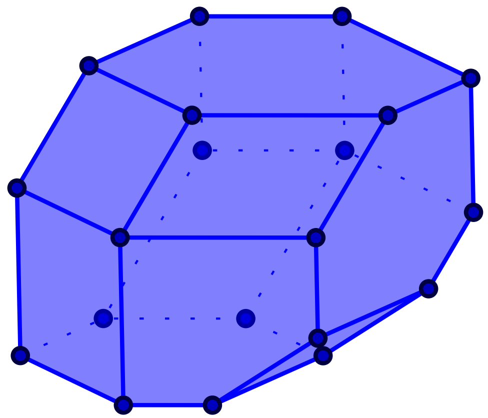
$$P^f_\equiv := \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; \langle\, \mathbf{r}(R) \mid \mathbf{x} \,\rangle \leq h^f_\equiv(R) \text{ for all } \varnothing \neq R \subsetneq [n] \right\}.$$

P.-Santos, *Quotientopes* ('17[+])

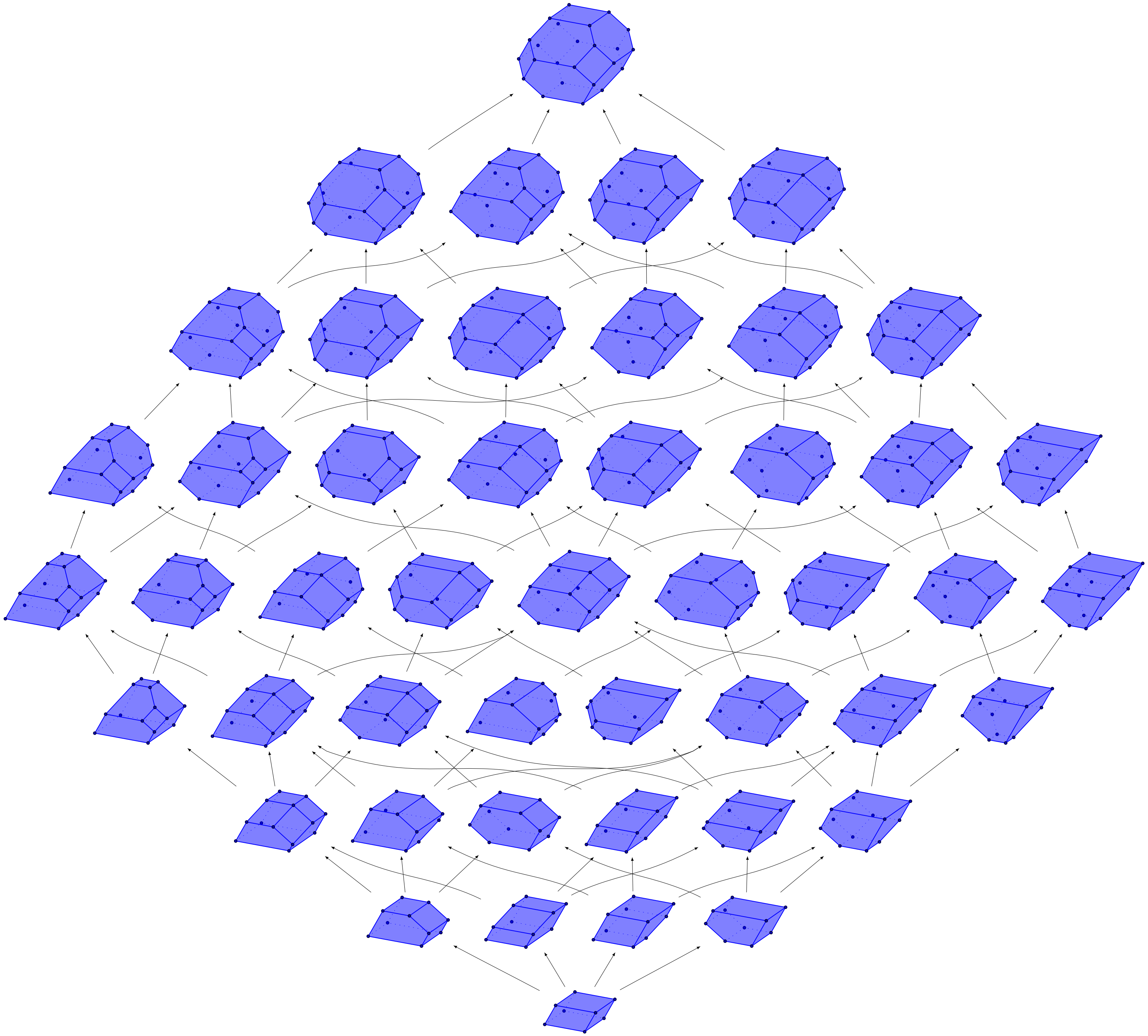




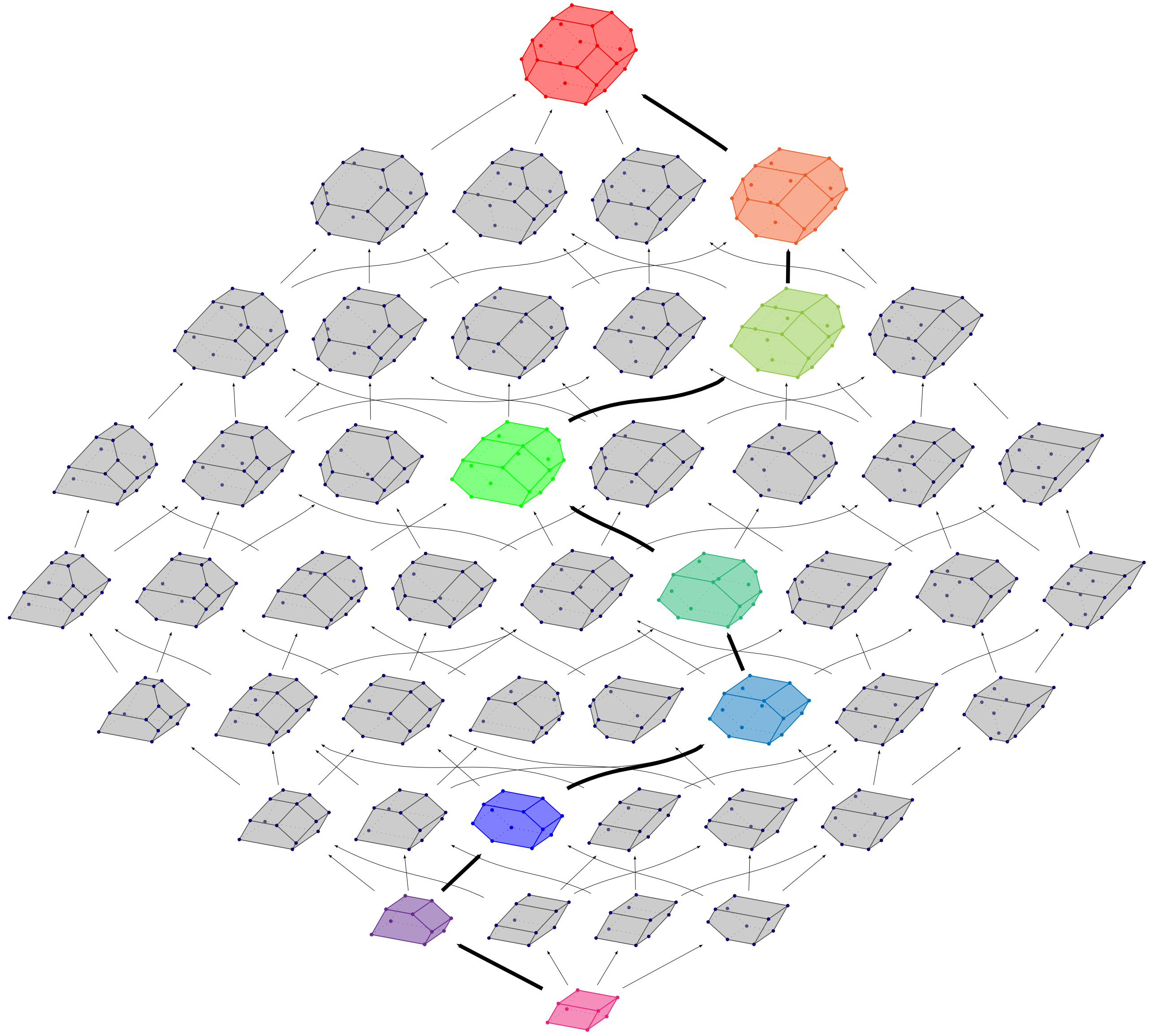




QUOTIENTOPE LATTICE



QUOTIENTOPE LATTICE



POLYWOOD

TOWARDS QUOTIENTOPES FOR HYPERPLANE ARRANGEMENTS

$\mathcal{H}$ hyperplane arrangement in $\mathbb{R}^n$
B distinguished region of $\mathbb{R}^n \smallsetminus \mathcal{H}$
inversion set of a region C = set of hyperplanes of $\mathcal{H}$ that separate B and C
poset of regions $\mathrm{Pos}(\mathcal{H}, B)$ = regions of $\mathbb{R}^n \smallsetminus \mathcal{H}$ ordered by inclusion of inversion sets

THM. The poset of regions $\mathrm{Pos}(\mathcal{H}, B)$

- is never a lattice when B is not a simple region,
- is always a lattice when $\mathcal{H}$ is a simplicial arrangement.

Björner-Edelman-Ziegler, *Hyperplane arrangements with a lattice of regions* ('90)

THM. If $\mathrm{Pos}(\mathcal{H}, B)$ is a lattice, and $\equiv$ is a lattice congruence of $\mathrm{Pos}(\mathcal{H}, B)$, the cones obtained by glueing together the regions of $\mathbb{R}^n \smallsetminus \mathcal{H}$ in the same congruence class form a complete fan.

Reading, *Lattice congruences, fans and Hopf algebras* ('05)

Is the quotient fan polytopal?