Algorithms for phase retrieval problems

Part I

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Journées de géométrie algorithmique
Aussois

Inverse problems

Inverse problems are problems of the form :

> Reconstruct an "unknown object"
> from some set of "measurements" ?

In this mini-course, we will discuss a family of inverse problems, called phase retrieval problems.

Outline

This morning : introduction to phase retrieval problems.

- Definition, motivations
- When is there hope to solve a phase retrieval problem ?
  - First case : "generic" problems
  - Second case : "non-generic" problems

This afternoon : a first family of reconstruction algorithms.

Tomorrow : a second family of reconstruction algorithms.

Phase retrieval problems : definition

Let

- V be a fixed complex vector space ;
- x be an unknown element of V ;
- $(L_s)_{s\in S}$ be a (known) family of linear forms on V.

The corresponding linear inverse problem would be :

Reconstruct $x \in V$ from $(L_s(x))_{s\in S}$?

Typically "easy" to solve.

Phase retrieval problems : definition

Here, we assume that we do not have access to the phase of $L_s(x)$.

$\rightarrow$ We have only its modulus.

General form of a phase retrieval problem :

$$\text{Reconstruct } x \in V \text{ from } (|L_s(x)|)_{s \in S}\,?$$

(Called "phase retrieval" because, if you can retrieve the phases, you can easily reconstruct x by solving the linear inverse problem.)

Global phase ambiguity

Reconstruct $x \in V$ from $(|L_s(x)|)_{s \in S}$?

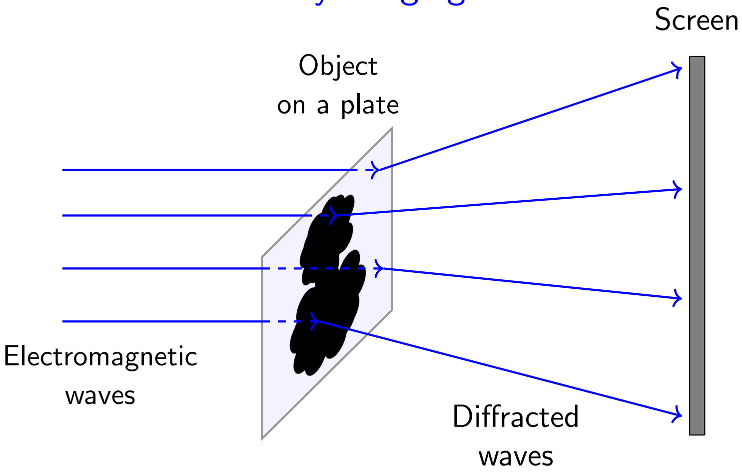
Let $u \in \mathbb{C}$ have modulus 1. For any $s \in S$,

$$\begin{aligned} |L_s(ux)| &= |uL_s(x)| \\ &= |L_s(x)|. \end{aligned}$$

$\Rightarrow$ It is not possible to distinguish x from ux.

We only aim at recovering x up to multiplication by a unitary complex number.
We call it reconstruction up to a global phase.

First motivation : X-ray imaging



[Schechtman, Eldar, Cohen, Chapman, Miao, and Segev, 2015]

First motivation : X-ray imaging

We aim at recovering the support function of the object :

$$
\begin{array}{rccll}
f : & \mathbb{R}^2 & \to & \mathbb{C} & \\
 & (x,y) & \to & 1 & \text{if point } (x,y) \text{ of the plate is masked by the object,} \\
 & (x,y) & \to & 0 & \text{otherwise.}
\end{array}
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$$

The diffracted wave, when it arrives to the screen, can be described by a function $F_{screen} : \mathbb{R}^2 \to \mathbb{C}$.

This F_{screen} is (approximately) the Fourier transform of f :

$$F_{screen} = \hat{f}$$

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For any $(x, y) \in \mathbb{R}^2$, it is possible to measure $|F_{screen}(x, y)|$, but not $\mathrm{phase}(F_{screen}(x, y))$.

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Reconstruct a compactly supported f from $(|\hat{f}(x, y)|)_{(x,y)\in\mathbb{R}^2}$?

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Reconstruct a compactly supported f from $(|\hat{f}(x, y)|)_{(x,y)\in\mathbb{R}^2}$?

This is a phase retrieval problem.
(The most important one, historically and in applications.)

Second motivation : audio processing

Goal of audio processing : develop algorithms to solve tasks involving sound recordings, that would be too difficult or tedious for human persons to perform.

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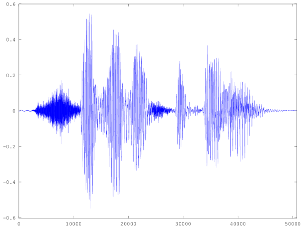
Main difficulty : an audio signal is a complicated object.

It can be represented by a function

$$f : \mathbb{R} \to \mathbb{R},$$

usually with no obvious "structure".

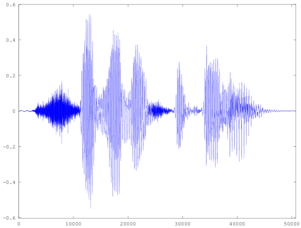


Second motivation : audio processing

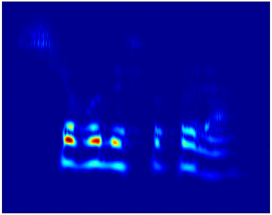
Idea : use an intermediate representation, easier to analyze.

$$(f : \mathbb{R} \to \mathbb{R}) \quad \longrightarrow \quad (|Wf| : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+).$$

$|Wf|$: modulus of the “short-time Fourier transform” or “wavelet transform” of f.



$f : \mathbb{R} \to \mathbb{R}$



$|Wf| : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$

Second motivation : audio processing

Typical pipeline of an algorithm

$$\boxed{f} \longrightarrow \boxed{|Wf|} \xrightarrow{\text{some analysis}} \boxed{\text{result}}$$

When the expected result is a new audio signal,

$$\boxed{f} \longrightarrow \boxed{|Wf|} \xrightarrow{\text{some analysis}} \boxed{|Wg|} \xrightarrow{\text{reconstruction}} \boxed{\text{g}}$$

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This step is a phase retrieval problem.

Reconstruct $x \in V$ from $(|L_s(x)|)_{s \in S}$?

When can we hope to solve this problem ?

At least two possible issues :

- No uniqueness of the reconstruction :

  There exist $x \neq y$ such that $\forall s, |L_s(x)| = |L_s(y)|$.

- No stability

  There exist $x \not\approx y$ such that $\forall s, |L_s(x)| \approx |L_s(y)|$.

If one of these issues arises, we say the the problem is ill-posed.

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Two main situations

1. The L_s are "generic" or random, chosen according to a simple probability law.
   - Analysis is doable.
   - Uniqueness and stability tend to hold.
2. The L_s are deterministic, imposed by a specific application.
   - Analysis is usually difficult or impossible.
   - Uniqueness and stability may fail to hold.

Generic / random case, in finite dimension

Reconstruct $x \in V$ from $(|L_s(x)|)_{s \in S}$?

To simplify, we assume $\dim(V)$ and $\mathrm{Card}(S)$ to be finite.
We can assume :

- $V = \mathbb{C}^n$;
- $S = \{1, \dots, m\}$;
- $\forall s, L_s = \langle ., f_s \rangle$, for some $f_s \in \mathbb{C}^n$.
  ($\langle ., . \rangle$ is the usual hermitian product.)

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"measurement vectors"

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We can assume :

- $V = \mathbb{C}^n$;
- $S = \{1, \dots, m\}$; → "measurement vectors"
- $\forall s, L_s = \langle ., f_s \rangle$, for some $f_s \in \mathbb{C}^n$.
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Theorem (Uniqueness for generic measurements)

If $m \geq 4n - 4$, then, for generic vectors $f_1, \dots, f_m$, uniqueness holds :

$$\forall x, y \in \mathbb{C}^n, \qquad (\forall k, |\langle x, f_k \rangle| = |\langle y, f_k \rangle|) \quad \Rightarrow \quad (x = y).$$

"Generic" means that the set of m-tuples $(f_1, \dots, f_m)$ for which the property holds is open and dense in $(\mathbb{C}^n)^m$.

[Balan, Casazza, and Edidin, 2006]
[Conca, Edidin, Hering, and Vinzant, 2015]

Reconstruct $x \in \mathbb{C}^n$ from $(|\langle x, f_k\rangle|)_{1\leq k\leq m}$?

Generic uniqueness : idea of proof

Principle : dimension counting.

$$\left(\text{Not all } x \text{ are uniquely determined by } (|\langle x, f_k\rangle|)_{1\leq k\leq m}. \right)$$

$$\iff \left(\begin{array}{l} \text{There exist } x, y \in \mathbb{C}^n \text{ such that} \\ \qquad \forall k \leq m, |\langle x, f_k\rangle| = |\langle y, f_k\rangle|. \end{array} \right)$$

$$\iff \left(\begin{array}{l} \text{There exist } x, y \in \mathbb{C}^n \text{ and } \phi_1, \dots, \phi_m \in \mathbb{R} \text{ such that} \\ \qquad \forall k \leq m, \langle x, f_k\rangle = e^{i\phi_k}\langle y, f_k\rangle. \end{array} \right)$$

Idea of proof (continued)

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$$\iff \left((f_1, \dots, f_m) \in \bigcup_{\substack{x,y \in \mathbb{C}^n \\ \phi_1,\dots,\phi_m \in \mathbb{R}}} \{x - e^{i\phi_1} y\}^\perp \times \dots \times \{x - e^{i\phi_m} y\}^\perp \right)$$

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Idea of proof (continued)

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The set of all $(f_1, \dots, f_m)$ for which uniqueness does not hold is a (union of) manifold(s), with dimension at most $2n + 2n + m + m(2n - 2) = 2mn + 4n - m$.

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The set of all $(f_1, \dots, f_m)$ for which uniqueness does not hold is a (union of) manifold(s), with dimension at most $2n + 2n + m + m(2n-2) = 2mn + 4n - m$.

$2mn + 4n - m < 2mn = \dim_{\mathbb{R}}((\mathbb{C}^n)^m)$ when $m \geq 4n + 1$.

Reconstruct $x \in \mathbb{C}^n$ from $(|\langle x, f_k \rangle|)_{1 \le k \le m}$?

When $m \geq 4n - 4$, generic measurements $\Rightarrow$ uniqueness.

Can we have uniqueness with less measurements than $4n - 4$?

- If $m < 4n - O(\log(n))$, uniqueness never holds. [Heinosaari, Mazzarella, and Wolf, 2013]
- For some n, when $m < 4n - 4$, uniqueness never holds. [Conca, Edidin, Hering, and Vinzant, 2015]
- For some n, uniqueness can be reached with $m < 4n - 4$. [Vinzant, 2015]

Summary

- $m \sim 4n$ measurements generically ensure uniqueness.
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What about stability ?

We are never given the exact value of $(|\langle x, f_k\rangle|)_{1\leq k\leq m}$.
In this case, can we do approximate reconstruction ?

For example, is reconstruction Lipschitz ?

$$\forall x, y \in \mathbb{C}^n, \quad ||x-y||_2 \leq C \left|\left| \begin{pmatrix} |\langle x, f_1\rangle| \\ \vdots \\ |\langle x, f_m\rangle| \end{pmatrix} - \begin{pmatrix} |\langle y, f_1\rangle| \\ \vdots \\ |\langle y, f_m\rangle| \end{pmatrix} \right|\right|_2 ?$$

(for some $C > 0$)

Stability

Theorem

When uniqueness holds, reconstruction is always Lipschitz.

[Balan and Zou, 2016]

Stability

Theorem

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[Balan and Zou, 2016]

Problem : the Lipschitz constant can be terrible.
In particular, it can depend on n and m.

→ Stronger assumptions on $f_1, \dots, f_m$?

We assume that the f_k are chosen at random, independently, according to normal distributions :

$$f_k \overset{i.i.d.}{\sim} \mathcal{N}_{\mathbb{C}}(0, \mathrm{Id}_n), \quad k = 1, \dots, m.$$

Theorem

There exist constants $\alpha, \gamma, C > 0$ such that, if

$$m \geq \alpha n,$$

reconstruction is C-Lipschitz with proba at least $1 - O(e^{-\gamma m})$.

[Candès and Li, 2014]

Summary

For generic/random f_k's, the problem is well-posed.

- Uniqueness holds "generically".
- Stability holds with high probability.

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For generic/random f_k's, the problem is well-posed.

- Uniqueness holds "generically".
- Stability holds with high probability.

But in applications, measurements are neither generic nor random.

$\rightarrow$ Is the phase retrieval problem still well-posed ?

Non-generic/random phase retrieval

Is the phase retrieval problem still well-posed ?

Answering this question is difficult.
Only a handful of particular cases can be precisely studied.

We will discuss the main one : the Fourier transform.

($\rightarrow$ X-ray imaging)

Phase retrieval for the Fourier transform

Let $x \in \mathbb{C}^n$ be a vector.
Its (semi)discrete Fourier transform is

$$\hat{x} : \omega \in [0; 2\pi[\quad \to \quad \hat{x}(\omega) = \sum_{k=0}^{n-1} x_k e^{-ik\omega}$$

The corresponding phase retrieval problem is

Reconstruct $x \in \mathbb{C}^n$ from $(|\hat{x}(\omega)|)_{\omega \in [0;2\pi[}$?

Reconstruct $x \in \mathbb{C}^n$ from $(|\hat{x}(\omega)|)_{\omega \in [0;2\pi[}$?

For all ω,
$$|\hat{x}(\omega)|^2 = \left| \sum_{k=0}^{n-1} x_k e^{-ik\omega} \right|^2 = \left(\sum_{k=0}^{n-1} x_k e^{-ik\omega} \right) \left(\sum_{k=0}^{n-1} \overline{x_k} e^{+ik\omega} \right) = P_x(e^{-i\omega}) \overline{P_x} \left(\frac{1}{e^{-i\omega}} \right),$$

where
$$P_x(X) = \sum_{k=0}^{n-1} x_k X^k.$$

Reconstruct $x \in \mathbb{C}^n$ from $(|\hat{x}(\omega)|)_{\omega\in[0;2\pi[}$?

$$\left(\text{Knowing } (|\hat{x}(\omega)|)_{\omega\in[0;2\pi[}\right)$$

$$\iff \left(\text{Knowing } P_x(X)\overline{P}_x(1/X)\right)$$

Let us write $P_x(X) = c\prod_{s=1}^{n-1}(X - z_s)$.

Then $P_x(X)\overline{P}_x(1/X) = \tilde{c}X^{-(n-1)}\prod_{s=1}^{n-1}(X - z_s)(X - \frac{1}{\overline{z_s}})$.

From $|\hat{x}|$, we can determine $z_1, \dots, z_{n-1}$
up to the transformation $z \to \frac{1}{\overline{z}}$.

Phase retrieval for the Fourier transform

Summary

From $(|\hat{x}(\omega)|)_{\omega\in[0;2\pi[}$, we can recover P_x (and hence x) up to a "flipping" of each root.

Since there are $n-1$ roots, for most vectors x,

there are 2^{n-1} vectors y such that $|\hat{x}| = |\hat{y}|$.

$\Rightarrow$ No uniqueness.

Phase retrieval for the Fourier transform

In higher dimensions

And in dimension $d \geq 2$?

We can define a "dimension d" Fourier transform, on vectors $x \in \mathbb{C}^{n^d}$, and look at the same phase retrieval problem.

The picture is different :

"Almost all" x are uniquely determined from $|\hat{x}|$.

But reconstruction is not Lipschitz.

Other problems than the Fourier transform ?

Other measurement systems can be analyzed, but they need to have special properties.

("Special properties" : typically, a relation with harmonic analysis that allows to use similar tools as for Fourier.)

For these other measurement systems, the problem is

- possibly "less ill-posed" than for Fourier ;
- typically "more ill-posed" than for random measurements.

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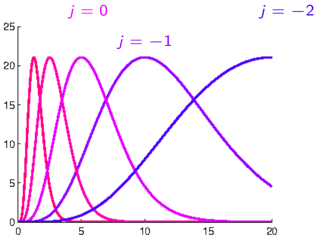
- typically "more ill-posed" than for random measurements.

→ We will briefly discuss the wavelet transform case.

Phase retrieval for the wavelet transform

$$\begin{array}{rccc} W: & L^2(\mathbb{R}) & \to & (L^2(\mathbb{R}))^{\mathbb{Z}} \\ & f & \to & Wf = (f \star \psi_j)_{j\in\mathbb{Z}}, \end{array}$$

where $(\psi_j)_{j\in\mathbb{Z}}$ is a (specific) family of band-pass filters.



Reconstruct $f \in L^2(\mathbb{R})$ from $|Wf| = (|f \star \psi_j|)_{j\in\mathbb{Z}}$?

Phase retrieval for the wavelet transform

To be able to analyze this problem, we need an assumption on the filters : we need them to be Cauchy wavelets.

Reason :
If the ψ_j's are Cauchy wavelets, then, for each j, $f \star \psi_j$ is the restriction onto a line of some specific holomorphic function.

Theorem (Mallat and Waldspurger [2015])

We assume the filters to be Cauchy wavelets.

- Uniqueness holds.
- Reconstruction is not Lipschitz, but "locally stable".

Phase retrieval for the short-time Fourier transform

$$\begin{array}{rccc} W: & L^2(\mathbb{R}) & \to & (L^2(\mathbb{R}))^{\mathbb{Z}} \\ & f & \to & Wf = (\widehat{w_n f})_{n\in\mathbb{Z}}, \end{array}$$

with w a compactly-supported window, w_n its translation by n.

Reconstruct f from $|Wf| = (|\widehat{w_n f}|)_{n\in\mathbb{Z}}$?

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> Reconstruct f from $|Wf| = (|\widehat{w_n f}|)_{n\in\mathbb{Z}}$?

For Gaussian windows, similar results as for wavelets :

- Uniqueness holds.
- Reconstruction is not Lipschitz, but "locally stable".

[Alaifari, Daubechies, Grohs, and Yin, 2016]
[Grohs and Rathmair, 2017]

Phase retrieval for the short-time Fourier transform

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with w a compactly-supported window, w_n its translation by n.

Reconstruct f from $|Wf| = (|\widehat{w_n f}|)_{n\in\mathbb{Z}}$?

For general windows,

- Almost all signals are uniquely determined.
- Stability unknown.

[Jaganathan, Eldar, and Hassibi, 2016]

Summary

- We have introduced a family of inverse problems.
- We have discussed two applications.
- We have discussed the well-posedness of these problems.
    - In a generic/random setting, well-posed.
    - In a non-generic setting, possibly ill-posed ; difficult to study.

This afternoon

- Focus on the generic/random setting.
- Discuss reconstruction algorithms.