Reach comparison theory

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Motivation

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Riemannian Simplices and triangulations

- Arbitrary Riemannian manifolds
- Used for triangulation

Open problems: submanifolds of Riemannian manifolds, manifolds with boundary and stratified manifolds:

- Natural extension
- Occur in many applications, such as dynamical systems

Previous results in Euclidean space:

- Submanifolds: mesh generation and reconstruction communities.
- Stratification of $\mathbb{R}^3$ (Rineau, Yvinec, Dey)

How to use results in Euclidean space?

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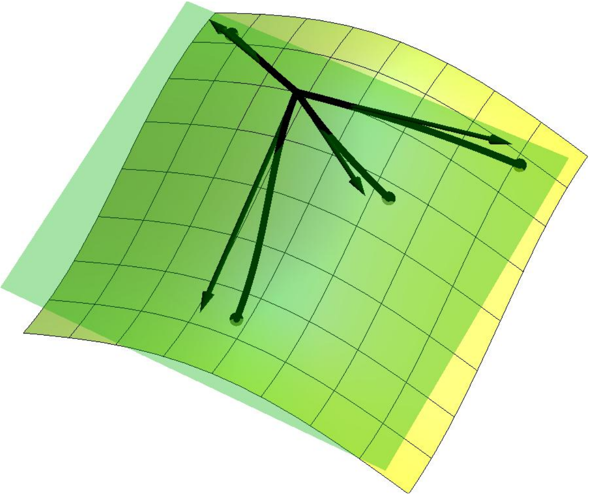
There is a natural way to locally go from the manifold to the tangent space (which is Euclidean) and back: the exponential map.



Final goal: triangulations

The approach we intend to follow is:

- Lift everything locally to the tangent spaces at a set of specific points
- Triangulate in the tangent space
- Use previous results to stitch the local triangulations together.

To triangulate in the tangent space we need to understand the geometry of the lifted submanifold, boundary, or strata, respectively, in particular we need to understand the reach.

Here we focus on the reach.

Outline

- The reach
- The Toponogov comparison theorem
- The proof of the Toponogov comparison theorem in 2D
- Extension of the proof to arbitrary dimension
- Higher order comparison theory in 2D

The reach

The reach

On a manifold we can define the reach in much the same way as in Euclidean space:
Distance to medial axis (some extra technical assumption because of the injectivity radius)

Balls tangent to a submanifold with radius less than the reach are empty.
Notice that geodesic balls are also Euclidean ball if lifted to the tangent space at their centre.

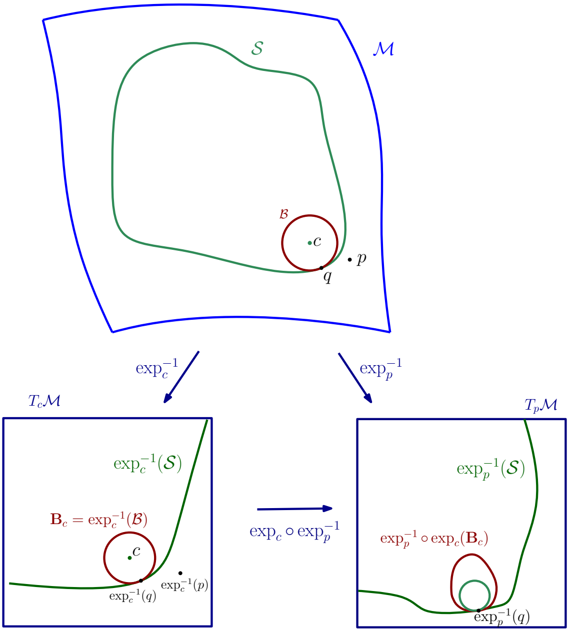
From the metric to a transformation

Simple example: From a Euclidean metric $\mathrm{d}x_1^2 + \ldots \mathrm{d}x_d^2$ to itself $\Rightarrow$ translation and rotation.

Suppose two metrics g and $\tilde{g}$ in two coordinate systems and both are not too far from Euclidean and bounds on the derivative of the metric then we can give bounds on the transformation from g to $\tilde{g}$, up to quadratic order.

A result by Federer gives a lower bound on the reach after a transformation assuming that you know it up to quadratic order.

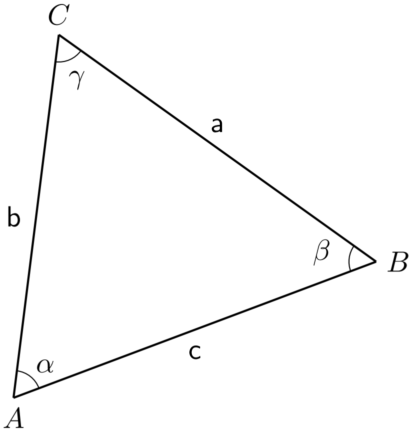
In our setting g and $\tilde{g}$ induced by $\exp_p^{-1}$ and $\exp_c^{-1}$ and the transformation is $\exp_p -1 \circ \exp_c^{-1}$.



$\mathcal{S}$
$\mathcal{M}$
$\mathcal{B}$
c
p
q
$\exp_c^{-1}$
$\exp_p^{-1}$
$T_c\mathcal{M}$
$T_p\mathcal{M}$
$\exp_c^{-1}(\mathcal{S})$
$\mathbf{B}_c = \exp_c^{-1}(\mathcal{B})$
c
$\exp_c^{-1}(q)$
$\exp_c^{-1}(p)$
$\exp_c \circ \exp_p^{-1}$
$\exp_p^{-1}(\mathcal{S})$
$\exp_p^{-1} \circ \exp_c(\mathbf{B}_c)$
$\exp_p^{-1}(q)$

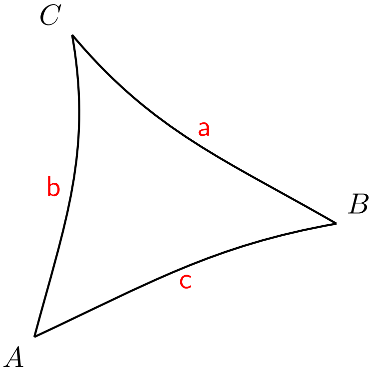
Topogonov comparison theorem

Triangles



Geodesic triangles

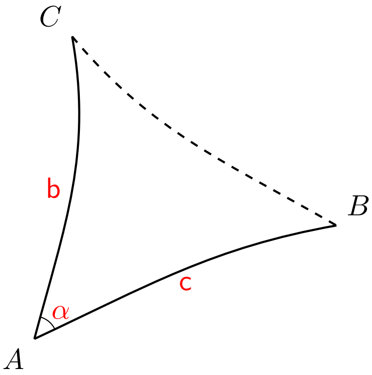
Geodesic triangle T: Three minimizing geodesics connecting three points on a arbitrary manifold (no interior)



Alexandrov triangle: A geodesic triangle with same edge lengths a, b, c on space of constant curvature $\mathbb{H}(\Lambda_*)$.

Hinges

Hinge: Two minimizing geodesics connecting three points and enclosed angle on a arbitrary manifold



Rauch hinge: A hinge with the same edge lengths b, c and enclosed angle α on a space of constant curvature $\mathbb{H}(\Lambda_*)$.

Topogonov comparison theorem

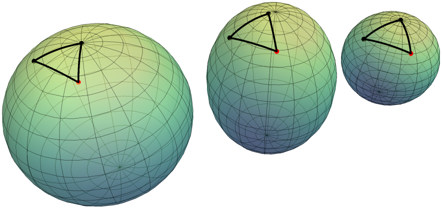
Manifold M, sectional curvatures $\Lambda_- \leq K \leq \Lambda_+$.

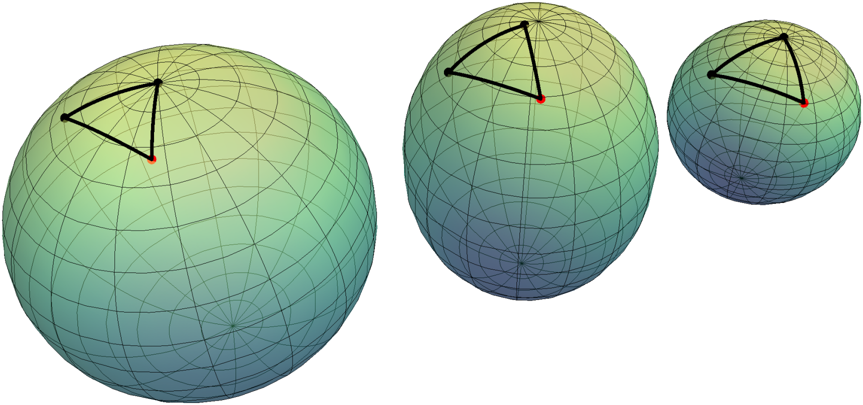
Given geodesic triangle T on M then exist Alexandrov triangles T_{Λ_-}, T_{Λ_+} on $\mathbb{H}(\Lambda_-)$, $\mathbb{H}(\Lambda_+)$ and

$$\alpha_{\Lambda_-} \leq \alpha \leq \alpha_{\Lambda_+}.$$

Given hinge on M then exist Rauch hinges on $\mathbb{H}(\Lambda_-)$, $\mathbb{H}(\Lambda_+)$ and the length of the closing geodesics satisfy

$$c_{\Lambda_-} \geq c \geq c_{\Lambda_+}.$$





The proof in 2D

Sketch of the proof in 2D for $\Lambda_- \leq K$ (Berger)

Lemma (Gauss)

Let $p \in M$*,* $v \in T_pM$ *and* $w \in T_pM \equiv T_vT_pM$

$$\langle (\mathrm{d}\exp_p)_v(v), (\mathrm{d}\exp_p)_v(w) \rangle = \langle v, w \rangle$$

- With Gauss lemma: normal coordinates (via $\exp$):

$$\mathrm{d}s^2 = \mathrm{d}r^2 + f^2(r,\theta)\mathrm{d}\theta^2 = \mathrm{d}r^2 + (r^2 + \mathcal{O}(r^2))\mathrm{d}\theta^2$$

- the Gauss (or sectional) curvature satisfies

$$f'' + Kf = \frac{\mathrm{d}^2 f}{\mathrm{d}r^2}(r,\theta) + K(r,\theta) f(r,\theta) = 0$$

We write

$$f''_{\Lambda_-} + \Lambda_- f_{\Lambda_-} = \frac{\mathrm{d}^2 f_{\Lambda_-}}{\mathrm{d}r^2}(r,\theta) + \Lambda_- f_{\Lambda_-}(r,\theta) = 0$$

Sketch of the proof in 2D for $\Lambda_- \leq K$ (Berger)

- $f'' = -Kf$ is the equation for a spring with initial conditions:

$$\mathrm{d}r^2 + f^2(r,\theta)\mathrm{d}\theta^2 = \mathrm{d}r^2 + (r^2 + \mathcal{O}(r^2))\mathrm{d}\theta^2$$

  force has a lower bound implies extension of the spring has a lower bound: $f(r,\theta) \geq f_{\Lambda_-}(r,\theta) > 0$.

- length of any curve $\gamma(t) = (r(t), \theta(t))$

$$\int \sqrt{(\partial_t r(t))^2 + f^2(\partial_t \theta(t))^2}\mathrm{d}t \geq \int \sqrt{(\partial_t r(t))^2 + f^2_{\Lambda_-}(\partial_t \theta(t))^2}\mathrm{d}t$$

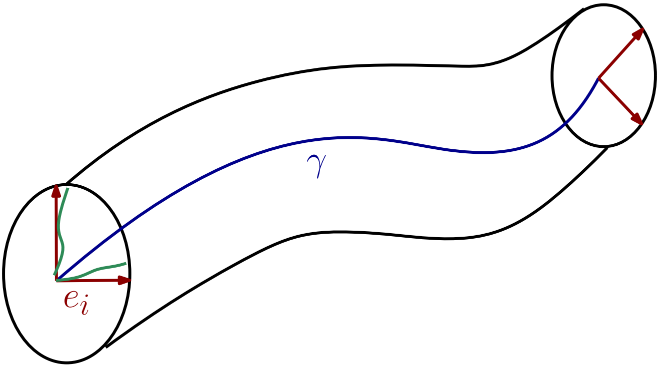
  for points $p = (r_0, \theta_0)$ and $q = (r_1, \theta_1)$

$$d_M(p,q) \geq d_{\Lambda_-}(p,q)$$

The extension to arbitrary dimension

Fermi coordinates

The construction of Fermi coordinates:



$$x = \exp_{\gamma(t)}\left(\sum_{i=1}^{d-1} x^i e_i(t)\right).$$

$(t = x^0, x^1, \ldots, x^{d-1})$ are coordinates for x.

We have for this coordinate system that:

Lemma (Manasse and Misner's lemma for Riemannian manifolds)

The metric in Fermi coordinates x along a geodesic γ is given by

$$g_{00}(x) = 1 + R_{0km0}x^k x^m + \mathcal{O}(|x|^3)$$
$$g_{0j}(x) = -\frac{2}{3}R_{kjm0}x^k x^m + \mathcal{O}(|x|^3)$$
$$g_{lj}(x) = \delta_{ij} - \frac{1}{3}R_{klmj}x^k x^m + \mathcal{O}(|x|^3)$$

From this we have the following corollary

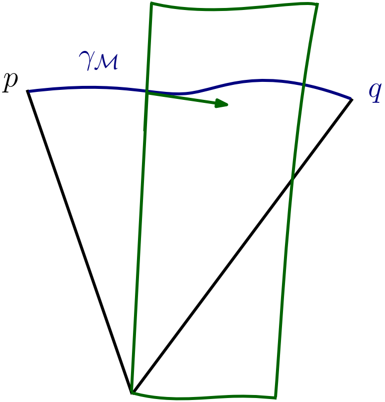
Corollary

Let $\mathcal{M}$ be a manifold whose sectional curvatures are bounded, that is $\Lambda_- \leq K \leq \Lambda_+$. If $x, y \in \mathcal{M}$ and $v \in T_y\mathcal{M}$, then there exists a surface S such that:

1. *The geodesic γ_{xy} connecting x and y is contained in S.*
2. $v \in T_y S$.
3. *The Gauss curvature K_S of S on the geodesic γ_{xy} is bounded by $\Lambda_- \leq K_S \leq \Lambda_+$.*

Using the analysis of the two dimensional case we now have

$$|v|_{\mathcal{M}} \geq |v|_{\Lambda_-}.$$



$$d_{\mathcal{M}}(p,q) = \int |\dot{\gamma}_{\mathcal{M}}(t)|_{\mathcal{M}} \mathrm{d}t$$
$$\geq \int |\dot{\gamma}_{\mathcal{M}}(t)|_{\Lambda_-} \mathrm{d}t \geq d_{\Lambda_-}(p,q).$$

Higher order comparison theory in 2D

Return to Berger's proof

$$\mathrm{d}s^2 = \mathrm{d}r^2 + f^2(r,\theta)\mathrm{d}\theta^2 = \mathrm{d}r^2 + (r^2 + \mathcal{O}(r^2))\mathrm{d}\theta^2$$

f determined by

$$f'' + Kf = \frac{\mathrm{d}^2 f}{\mathrm{d}r^2}(r,\theta) + K(r,\theta)f(r,\theta) = 0$$

We want not only bounds the metric, or equivalently f, but also its derivatives.

- The derivative with respect to r follows from Berger's proof.
- The derivative with respect to θ relies on perturbation theory

Perturbation theory

We return to the differential equation

$$f_\eta'' + (K - \eta B^{\partial K}) f_\eta = 0.$$

We write the solution as a series

$$f_\eta = f_0 + \eta f_1 + \eta^2 f_2 + \dots,$$

where the f_is are independent of η. Inserting our expansion in η into the differential equation, we find

$$\begin{aligned} f_0'' + K f_0 &= 0 \\ f_1'' - B^{\partial K} f_0 + K f_1 &= 0. \end{aligned}$$

One can then proceed to give bounds on f_1.
General case due to Kaul.

Questions?